LECTURE 1: THE FIRST BETTI NUMBER OF A COMPACT HYPERBOLIC MANIFOLD AND THE HODGE CONJECTURE FOR COMPACT QUOTIENTS OF THE COMPLEX n BALL

JOHN MILLSON

In 1976 in [10], I showed that for every n the standard arithmetic examples of compact hyperbolic n dimensional manifolds M had nonzero first Betti number by constructing nonseparating totally geodesic hypersurfaces inside them. Now, 36 years later, Nicolas Bergeron, Colette Moeglin and I can show, [2], that if $n \geq 5$ these totally-geodesic hypersurfaces *span* the next-to-top homology of M. This is proved by combining three very different pieces of work. First, there is the work I did with Steve Kudla in the 1980's (see the reference [8] below) where we constructed closed one forms which are Poincaré dual to the totally geodesic hypersurfaces using the Weil representation (the "special theta lift"). Second, there is the work of Kudla and Rallis, see for example [9], and Ginzburg, Jiang and Soudry, see [5] on the Siegel-Weil formula. Third, there is the very deep work of Jim Arthur, see for example [1], on the stabilized trace formula. Our proof procedes by first proving that a first cohomology class that comes from any Weil representation construction (any theta lift) comes from the special theta lift of [8] and therefore is necessarily in the subspace of the first cohomology spanned by the Poincaré duals of totally geodesic hypersurfaces. The last two pieces of work are then used to prove that *all* the first cohomology comes from some Weil representation construction (theta lift).

There are corresponding results showing that the Poincaré duals of certain totally geodesic cycles, which we will call "special cycles", span a definite part (a *refined Hodge component*) of the cohomology of the locally symmetric spaces of standard arithmetic type associated to the orthogonal groups $O(p,q)$. The notion of "refined Hodge type" introduced by Chern in [4] is closely related to the classification by David Vogan and Gregg Zuckerman, see [11], of the irreducible unitary representations π with nonzero relative Lie algebra cohomology (the refined Hodge type corresponds to the minimal K-type of π). The notion of refined Hodge type and the work of [11] play a key role in our work.

In very recent work (in progress) Bergeron, Moeglin and I have applied analogous techniques to the standard arithmetic quotients $M = \Gamma\backslash X$ of of the symmetric spaces X associated to the groups $U(p,q)$. For the case in which the unitary group is $U(n,1)$ the associated symmetric space X is the complex n ball D^n (complex hyperbolic space). We prove that, under the assumption $k < n/3$, the intersection $H^{2k}(M, \mathbb{Q}) \cap H^{k,k}(M, \mathbb{C})$ is spanned by the images in M of totally geodesic $n - k$-balls $D^{n-k} \subset D^n$ (these are the "special cycles" for this case). Since these image cycles are carried by projective subvarieties, *this proves the Hodge conjecture* in these degrees for the standard arithmetic quotients of the ball. The structure of the proof is the same as for the orthogonal case; however, for Part 2 it was necessary to prove the analogue of [5] for the unitary groups and for Part 3 it was necessary to extend results of Arthur to the unitary groups.

I will try to avoid technical details and explain the overall principles: the simple geometric idea behind my 1976 paper and the geometry behind the construction in [8] which uses the Weil representation to construct closed differential forms on the above manifolds which are Poincaré dual to the geodesic cycles.

REFERENCES

[1] J. Arthur *An introduction to the trace formula*, In *Harmonic analysis, the trace formula and Shimura varieties*, Clay Math. Proc. 4(2005), 1 -263, Amer. Math. Soc., Providence, RI.

[2] N. Bergeron, J. Millson and C. Moeglin , *Hodge type theorems for arithmetic manifolds asociated to orthogonal groups* , preprint, arXiv:1110.3049.

[3] N. Bergeron, J. Millson and C. Moeglin , *On the Hodge conjecture for Shimura varieties associated to unitary groups* , in preparation.

[4] S. S. Chern , *On a generalization of Kähler geometry* , In *Algebraic geometry and topology. A symposium in honor of S. Lefschetz*, 103-121. Princeton University Press, Princeton, N.J. 1957.

[5] D. Ginzburg, Dihua Jiang and D. Soudry , *Poles of L-functions and theta-liftings for orthogonal groups* , J. Inst. Math. Jussieu, **8** (2009), 693-741.

[6] D.Kazhdan , *Connection of the dual space of a group with the structure of its closed subgroups* , Func. Anal. Appl **1** (1967), 63-65.

[7] D.Kazhdan , *Some applications of the Weil representation* , J. Anal. Math. **32** (1977), 235-248.

[8] S.Kudla and J. Millson , *Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables*, Inst. Hautes Études Sci. **71** (1990), 121-172.

[9] S.Kudla and S. Rallis , *A regularized Siegel-Weil formula: the first term identity*, Annals of Math . **140** (1994), 1-80.

[10] J. Millson , *On the first Betti number of a constant negatively curved manifold* , Annals of Math. **104** (1976), 235-247.

[11] D. Vogan and G. Zuckerman , *Unitary representations with nonzero cohomology* , Compositio Math. . **53** (1984), 51-90.

LECTURE 2: THE GENERALIZED TRIANGLE INEQUALITIES IN SYMMETRIC SPACES AND EUCLIDEAN BUILDINGS WITH APPLICATIONS TO ALGEBRA

In my second lecture, I will talk about results I proved with Bernhard Leeb and Misha Kapovich in a series of papers - see [KLM1],[KLM2] and [KLM3] about the *triangle inequalities* for higher rank symmetric spaces $X = G/K$ of noncompact type and their relations to the structure constants for the spherical Hecke algebra of G and the representation ring of the Langlands dual group $G^\vee$. These results were complemented by the paper [KM2] which generalized the famous saturation theorem of Knutson and Tao for $\mathrm{GL}(n)$ to any simple reductive group G (except one needs the saturation factor k_G, see below). For a good exposition of the background to this lecture see the expository article of Fulton [Fu] in the Bulletin of the AMS.

The eigenvalues of a sum problem. In a rank one symmetric space of noncompact type $X = G/K$ the only invariant (under isometry) of pairs of points x, y is the distance $d(x, y)$ between them. In a symmetric space of rank k there are k real-valued invariants. The scalar-valued distance $d(x, y)$ gets replaced by a *cone-valued distance* which assigns to a pair of points $x, y \in X$ a point $d_\Delta(x, y)$ in the Weyl chamber Δ, a k-dimensional cone in the Cartan subspace $\mathfrak{a} = \mathbb{R}^k$. Given three points $\alpha, \beta, \gamma \in \Delta$ one may ask for conditions on α, β, γ that are necessary and sufficient in order that one can draw a geodesic triangle in X with side-lengths α, β, γ, precisely find three points $x, y, z \in X$ so that $d_\Delta(x, y) = \alpha, d_\Delta(y, z) = \beta, d_\Delta(z, x) = \gamma$. The answer is that there is a system of homogeneous linear inequalities determined by the Schubert calculus in the Grassmanians G/P for P a maximal parabolic subgroup of G) that give these necesssary and sufficient conditions. For the case of $G = \mathrm{GL}(n, \mathbb{C})$ the triangle inequalities specialize to the famous inequalities discovered by Klyachko [Kly1] which solve the following classical problem which dates back to Hermann Weyl (see [Fu] for the history)

Problem. Given two Hermitian matrices A and B with eigenvalues $\lambda_1, \cdots, \lambda_n$ and $\mu_1, \cdots, \mu_n$ respectively what are the possible eigenvalues for the sum $A + B$.

The inequalities of Klyachko are irredundant but for groups G other than $G = \mathrm{GL}(n, \mathbb{C})$ the generalized triangle inequalities are highly redundant, see for example [KuLM] where the inequalities were computed for all the rank three cases. A much smaller subset of the generalized triangle inequalities with the same set of solutions was found by P. Belkale and S. Kumar in [BK] and the resulting subsystem was proved to be the irredundant one by Ressayre in [Re].

The connection with decomposing tensor products of finite dimensional representations. Now suppose that $G = G(\mathbb{R})$ is the group of real points of a reductive algebraic group G defined over $\mathbb{Z}$. Then we have the p-adic group $G(\mathbb{Q}_p)$ and the Langlands' dual group $G^\vee$. Suppose further that α, β, γ are integral in the sense that they are cocharacters of a maximal torus T of G defined over $\mathbb{Z}$. Such cocharacters λ parametrize basis elements f_λ for the spherical Hecke algebra $\mathcal{H}(G(\mathbb{Q}_p))$ of $G(\mathbb{Q}_p)$ and basis elements ch_λ of the representation ring $\mathcal{R}(G^\vee$ of $G^\vee$. The Satake isomorphism gives an explicit isomorphism $\mathcal{H}(G(\mathbb{Q}_p)) \cong \mathcal{R}(G^\vee)$. However the two bases are not compatible with the isomorphism (the subtle relation between the two bases was computed by Luzstig [Lu]). Thus, using the

Satake isomorphism to identify $\mathcal{H}(G(\mathbb{Q}_p))$ and $\mathcal{R}(G^\vee)$ we have *two different bases for the same algebra* and it makes sense to compare the two sets of resulting structure constants.

Accordingly, we define two sets of triple structure constants $m(\,\cdot\,,\,\cdot\,,\,\cdot\,)$ and $n(\,\cdot\,,\,\cdot\,,\,\cdot\,)$ parametrized by triples of dominant cocharacters by

$$f_\alpha \bullet f_\beta \bullet f_\gamma = m(\alpha, \beta, \gamma)1 + \cdots \text{ and } ch_\alpha \bullet ch_\beta \bullet ch_\gamma = n(\alpha, \beta, \gamma)1 + \cdots .$$

Let k_G be the LCM of the coefficients of the highest root when expressed in terms of the simple roots. Assume $\alpha + \beta + \gamma$ is in the coroot lattice. Then we have the following

Theorem.
  (1) $n(\alpha, \beta, \gamma) \neq 0 \Rightarrow m(\alpha, \beta, \gamma) \neq 0 \Rightarrow \alpha, \beta, \gamma$ satisfy the triangle inequalities.
  (2) α, β, γ satisfy the triangle inequalities $\Rightarrow m(k_G\alpha, k_G\beta, k_G\gamma) \neq 0 \Rightarrow n(k_G^2\alpha,\ k_G^2\beta, k_G^2\gamma) \neq 0$.

For $GL(n)$ we have $k_G = 1$ so the previous theorem includes the saturation theorem of Knutson and Tao, [KT]. For the other classical groups $k_G = 2$. Considerable effort has gone into improving the "saturation factor" k_G^2 for the simple groups other than $GL(n)$. There is still a long way to go. For example, $k_{E_8} = 60$ and at this date the only known saturation factor for E_8 is that of [KM2] namely $k_{E_8}^2 = (60)^2 = 3600$. Kapovich and Millson,[KM1], have conjectured that for E_8 statement (2) of the previous theorem remains true with 3600 replaced by *one*.

REFERENCES

[BK] P. Belkale and S. Kumar, *Eigenvalue problem and a new product in cohomology of flag varieties*, Invent. Math. **166** (2008), 185–228.

[Fu] W. Fulton, *Eigenvalues, invariant factors, highest weights, and Schubert calculus*, Bull. Amer. Math. Soc. **37** (2000), 209–249.

[KKM] M. Kapovich, S. Kumar and J. J. Millson, *The eigencone and saturation for Spin(8)*, Pure and Applied Mathematics Quarterly **5**, no. 2 (2009) (Special Issue: In honor of Friedrich Hirzebruch, Part 1 of 2), 1–25.

[KLM1] M. Kapovich, B. Leeb and J. Millson, *Convex functions of symmetric spaces, side lengths of polygons and the stablitity inequalities for weighted configurations at infinity*, Journ. of Diff. Geom. **81** (2009),297–354.

[KLM2] M. Kapovich, B. Leeb and J. Millson, *Polygons in buildings and their refined side lengths*, Geom. Funct. Anal. **19** (2009), 1081–1100.

[KLM3] (with B. Leeb and M. Kapovich) *The generalized triangle inequalities in symmetric spaces and buildings with applications to algebra*, Memoirs of the AMS, Vol. 192, No. 896 (2008).

[KM1] M. Kapovich and J. Millson, *Structure of the tensor product semigroup*, Asian J. Math. **10**, no. 3 (2006)(papers dedicated to the memory of S.S. Chern), 493–540.

[KM2] M. Kapovich and J. Millson, *A path model for geodesics in Euclidean buildings and applications to representation theory*, Groups, Geometry and Dynamics 2 (2008),pp. 405-480.

[Kly1] A. Klyachko, *Stable bundles, representation theory and Hermitian operators*, Selecta Mathematica **4** (1998), 419–445.

[KT] A. Knutson and T. Tao, *The honeycomb model of* $GL_n(\mathbb{C})$ *tensor products I. Proof of the saturation conjecture*, J. Amer. Math. Soc., **12** (1999), no. 4, p. 1055–1090.

[KTW] A. Knutson, T. Tao and C. Woodward, *The honeycomb model of* $GL_n(\mathbb{C})$ *tensor products II: Puzzles determine the factes of the Littlewood-Richardson cone*, J. Amer. Math.Soc **17** (2004), no. 1, p. 19–48.

[KuLM] S. Kumar,B. Leeb and J. Millson, *The generalized triangle inequalities for rank 3 symmetric spaces of noncompact type*, Contemporary Math. **332** (2003) (volume dedicated to Robert Greene), 171–195.

[Lu] G. Lusztig, *Singularities, character formulas and a q-analogue of weight multiplicities* , Astérisqe **101** (1983), 208–227.

[Re] N. Ressayre, *Geometric invariant theory and the generalized eigenvalue problem*, Invent. Math. **180**(2010), 389–441.

LECTURE 3: THE TORIC GEOMETRY OF TRIANGULATED POLYGONS IN EUCLIDEAN SPACE

In my third lecture, I will talk about a relation between the moduli space $M_{\mathbf{r}}$ of n-gons with fixed side-lengths in $\mathbb{R}^3$, integrable systems of bending flows on $M_{\mathbf{r}}$ and flat toric degenerations $M_{\mathbf{r}}^{\mathcal{T}}$ of $M_{\mathbf{r}}$ attached to triangulations $\mathcal{T}$ of a reference planar convex n-gon.

In the paper [7], Misha Kapovich and I (and independently Klyachko in [8]) studied the space of of oriented congruence classes $M_{\mathbf{r}}$ of n-gons with fixed side-lengths $\mathbf{r} = (r_1, r_2, \cdots, r_n)$ in Euclidean three space. We showed that if $\mathbf{r}$ does not satisfy a certain fixed finite set of linear equations then $M_{\mathbf{r}}$ is a Kähler manifold. Also, in case the side lengths $(r_1, r_2, \cdots, r_n)$ are integers then the space $M_{\mathbf{r}}$ coincides with the projective variety which is the moduli space of n weighted (by $\mathbf{r}$) points on the projective line $\mathbb{CP}^1$. In what follows we will think of an n-gon as a n-tuple of vectors $\mathbf{e} = (e_1, \cdots, e_n)$ in Euclidean three space such that e_i has length r_i for $1 \leq i \leq n$ and the n-gon "closes up" in the sense that $e_1 + \cdots + e_n = 0$.

One of the main points of [7] and [8] was to construct a finite family of completely integrable systems on the space $M_{\mathbf{r}}$. There is one such family for each triangulation $\mathcal{T}$ of a reference planar convex n-gon π_n. The Hamiltonians in the system corresponding to $\mathcal{T}$ are the lengths of the diagonals used to make the triangulation $\mathcal{T}$. The Hamiltonian flow associated to a diagonal d (starting at a given n-gon $\mathbf{e}$) divides the n-gon $\mathbf{e}$ along d, rotates half of $\mathbf{e}$ in space around d at speed 1 and leaves the other half of $\mathbf{e}$ fixed. This led Kapovich and me to call the flows "bending flows". Note that if d is the zero vector then we don't know how to rotate around it and the corresponding bending flow is not defined. Because of this the space $M_{\mathbf{r}}$ is rarely toric even though it is "trying to be toric" in the sense it has $\frac{1}{2} \dim(M_{\mathbf{r}})$ commuting periodic Hamiltonian flows. The space $M_{\mathbf{r}}$ is stratified according to the vanishing of diagonals.

In [5] the authors suggested a way around the problem by pointing out that if one collapses parts of the above strata (creating a more singular stratified symplectic space) then the bending flows are everywhere defined in the collapsed space. We gave the name "$\mathcal{T}$-congruence" to the resulting coarsening of the equivalence relation of oriented congruence.

The construction of [5] was of particular interest because in [10] Speyer and Sturmfels had just constructed toric degenerations $M_{\mathbf{r}}^{\mathcal{T}}$ of the spaces $M_{\mathbf{r}}$ using combinatorial commutative algebra by weighting the Plücker coordinates for $\mathrm{Gr}(2, \mathbb{C})$ *using a weighting that depended on a triangulation $\mathcal{T}$ as above*. Thus it was a natural conjecture (made explicitly by Foth and Hu), [3], that **the stratified symplectic spaces constructed in [5] were isomorphic to those underlying the toric varieties of** [10].

The conjecture was proved in [4] for all triangulations $\mathcal{T}$. The point of my lecture will be to explain the proof. I believe that the examination of *singular* toric varieties as stratified symplectic spaces (this later notion has been developed by Reyer Sjamaar, see for example [9]), will be important in future developments. This is consistent with the overall theme of mirror symmetry which is to relate algebraic geometry and symplectic geometry. I intend to present an elementary introduction to toric varieties.

I am currently working with Chris Manon to generalize the above results to the moduli space of n weighted points on $\mathbb{CP}^m$. The integrable system(s) (the "bending flows") have already been constructed in [2] and the toric degenerations can be constructed using [1].

REFERENCES

[1] P.Caldero , *Toric degeneration of Schubert varieties* , Transformation Groups **7** (2002), pp. 51-60.

[2] H. Flaschka and J. Millson, *Bending flows for sums of rank one matrices* , Canad. J. of Math. **57** (2005),114-158.

[3] P. Foth and Y. Hu, *Toric degeneration of weight varieties and applications* , Trav. Math. XVI (2005), 87-105.

[4] B. Howard, C. Manon and J. Millson, *The toric geometry of triangulated polygons in Euclidean space* , Canad. J. of Math. **63** (2011), 878-937.

[5] T. Kamiyama and T. Yoshida, *Symplectic toric space associated to triangle inequalities*, Geom. Dedicata **93** (2002), 25-36. (2004)

[6] M. Kapovich and J. Millson, *On the moduli space of polygons in the Euclidean plane*, J. Diff. Geom. **42** (1995), 430-464.

[7] M. Kapovich and J. Millson, *The symplectic geometry of polygons in Euclidean space*, J. Diff. Geom. **44** (1996), 479-513.

[8] A. Klyachko, *Spatial polygons and stable configurations of points on the projective line* , Algebraic geometry and its applications (Yaroslavl,1992), Aspects Math., E25, Vieweg, Braunschweig, 1994, 67-84.

[9] R. Sjamaar and E. Lerman, *Stratified symplectic spaces and reduction*, Ann. of Math. **134** (1991), 375-422.

[10] D.Speyer and B. Sturmfels, *The tropical Grassmannian* Adv. Geom. **4** (2004),389-411.