Reductions of Kinetic Equations to Finite Component Systems

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Vlasov (Collisionless Boltzmann) Kinetic Equation

\[ \lambda_t + p\lambda_x - \lambda_p u_x = 0, \]

where (\( \lambda \) is a distribution function, \( x \) is a space coordinate, \( t \) is a time variable, \( p \) is a momentum)

\[ u = \int_{-\infty}^{\infty} \lambda dp. \]

Following the approach developed in plasma physics, one can introduce an infinite set of moments

\[ A^k = \int_{-\infty}^{\infty} p^k \lambda dp. \]

The Benney hydrodynamic chain is

\[ A_t^k + A_{x}^{k+1} + kA^{k-1}A_x^0 = 0, \quad k = 0, 1, 2, \ldots \]
Kinetic Equation Describing High Frequency Waves in Electron Plasma

\[ \lambda_t + p\lambda_x - E\lambda_p = 0, \]

where

\[ E_x = 1 - \int_{-\infty}^{\infty} \lambda dp, \quad E_t = \int_{-\infty}^{\infty} p\lambda dp. \]

Introducing moments of the distribution function

\[ A^k = \int_{-\infty}^{\infty} p^k\lambda dp, \]

the collisionless kinetic equation can be re-written as the nonlocal chain

\[ A_t^k + A_x^{k+1} + kA^{k-1}E = 0, \quad k = 0, 1, 2, \ldots, \]

where

\[ dE = (1 - A^0)dx + A^1dt \]

follows from (32).
Waterbag Reduction

\[ A^k = \frac{1}{k+1} \sum_{n=1}^{N} \epsilon_n (a^n)^{k+1}, \quad \sum_{n=1}^{N} \epsilon_n = 0, \]

where \( \epsilon_i \) are arbitrary constants.

Then Benney hydrodynamic chain reduces to the form

\[ a_i^t + \left( \frac{(a^i)^2}{2} + \sum_{n=1}^{N} \epsilon_n a^n \right)_x = 0, \]

where \( N \) is an arbitrary positive integer.

Nonlocal chain reduces to the nonlocal \( N \) component system

\[ a_i^t + a^i a_x^i + E = 0, \quad dE = (1 - A^0)dx + A^1 dt. \]
The nonlocal chain

\[ A_t^k + A_x^{k+1} + kA^{k-1}E = 0, \quad k = 0, 1, 2, \ldots, \]

where

\[ dE = (1 - A^0)dx + A^1dt \]

can be written in the evolution form

\[ E_t = A^1, \quad A_t^1 + A_x^2 + E(1 - E_x) = 0, \]

\[ A_t^k + A_x^{k+1} + kA^{k-1}E = 0, \quad k = 2, 3, \ldots, \]

where \( A^0 = 1 - E_x \). In such a form this non-hydrodynamic chain has just one local conservation law of Energy

\[ (A^2 + E^2)_t + A_x^3 = 0. \]
Local Representation

The nonlocal system

\[ a_t^k + a^k a_x^k + E = 0, \quad dE = (1 - A^0) dx + A^1 dt \]
transforms to \( N \) component evolution system \((k = 2, 3, \ldots, N)\)

\[ a_t^k + a^k a_x^k + E = 0, \]

\[ E_t = \frac{1}{2} \sum_{n=2}^{N} \epsilon_n (a^n)^2 + \frac{1}{2\epsilon_1} \left( E_x + \sum_{n=2}^{N} \epsilon_n a^n - 1 \right)^2, \]

where

\[ a^1 = \frac{1}{\epsilon_1} \left( 1 - \sum_{n=2}^{N} \epsilon_n a^n - E_x \right). \]
The waterbag reduction can be written in the Hamiltonian form

\[ E_t = - \sum_{n=2}^{N} \frac{\delta H}{\delta a^n}, \quad a^i_t = \frac{1}{\epsilon_i} \partial_x \frac{\delta H}{\delta a^i} + \frac{\delta H}{\delta E}, \quad i = 2, 3, ..., N, \]

which is determined by the local Hamiltonian

\[ H = - \frac{1}{2} \int \left[ E^2 + \frac{1}{3} \sum_{n=2}^{N} \epsilon_n (a^n)^3 + \frac{1}{3 \epsilon_1^2} \left( E_x + \sum_{n=2}^{N} \epsilon_n a^n - 1 \right)^3 \right] dx, \]

the momentum

\[ P = \int \left[ \frac{1}{2} \sum_{n=2}^{N} \epsilon_n (a^n)^2 + \frac{1}{2 \epsilon_1} \left( \sum_{n=2}^{N} \epsilon_n a^n \right)^2 + \frac{E_x}{\epsilon_1} \sum_{n=2}^{N} \epsilon_n a^n + \frac{E_x^2}{2 \epsilon_1^2} \right] dx \]

and \( N - 2 \) parametric Casimir

\[ Q = \int \sum_{n=2}^{N} \tilde{\epsilon}_n a^n, \quad \sum_{n=2}^{N} \tilde{\epsilon}_n = 0. \]
Periodic Solutions

Each field variable \( a^k \) depend on a sole phase \( \theta = x - \omega t \). Then the nonlocal system

\[
a^k_t + a^k a_x^k + E = 0, \quad dE = (1 - A^0)dx + A^1 dt
\]

reduces to an ODE system (\( \xi \) and \( \beta_k \) are arbitrary constants)

\[
u^2 = \frac{\xi}{u^2} - \delta + \frac{2u}{3} \epsilon_1 \pm \frac{2}{3u^2} \sum_{k=2}^{N} \epsilon_k (u^2 + \beta_k^2)^{3/2},
\]

where

\[
a^1 = u + \omega, \quad a^k = \omega \pm [u^2 + \beta_k^2]^{1/2}, \quad k = 2, 3, \ldots, N,
\]

This is \( N \) parametric periodic solution, whose characteristic velocity is determined by

\[
\omega = -\frac{1}{2\delta} \sum_{k=2}^{N} \epsilon_k \beta_k^2,
\]

which follows from comparison both above expressions for \( E' \).
Periodic Solutions

Since \( E = -uu' \), its explicit expression is

\[
E = -\sqrt{\xi - \delta u^2 + \frac{2u^3}{3} \epsilon_1 \pm \frac{2}{3} \sum_{k=2}^{N} \epsilon_k (u^2 + \beta_k^2)^{3/2}}.
\]

Then a corresponding solution of the distribution function \( \lambda(x, t, p) \) can be found due to the inverse formula

\[
\lambda(x, t, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iqp} \sum_{k=0}^{\infty} \frac{(-iq)^k}{k!} A^k dq,
\]

where

\[
A^k = \frac{1}{k+1} \sum_{n=2}^{N} \epsilon_n [(\omega \pm [u^2 + \beta_n^2]^{1/2})^{k+1} - (u + \omega)^{k+1}].
\]
N component Kodama reductions

\[ a^0_t + a^0_x + E = 0, \quad E_x = \delta + a^N, \quad a^k_t + \frac{1}{2} \left( \sum_{m=0}^{k} a^m a^{k-m} \right) = 0, \quad k = 1, \ldots, N \]

possess the local Hamiltonian structure

\[ a^0_t = -\frac{\delta H(N)}{\delta E}, \quad E_t = \frac{\delta H(N)}{\delta a^0}, \quad a^k_t = \partial_x \frac{\delta H(N)}{\delta a^{N-k}}, \quad k = 1, 2, \ldots, N - 1, \]

where the Hamiltonian is given by

\[ H(N) = \frac{1}{2} \int \left[ E^2 + (a^0)^2 (\delta - E_x) - \frac{a^0}{3} \sum_{k=1}^{N-1} a^k a^{N-k} - \frac{1}{3} \sum_{n=1}^{N-1} \sum_{k=0}^{n} a^k a^{n-k} a^{N-n} \right] dx \]

and the Momentum is

\[ P = \int \left[ \frac{1}{2} \sum_{k=1}^{N-1} a^k a^{N-k} + a^0 E_x \right] dx. \]
Periodic Solutions

Each field variable $a^i$ depend on a sole phase $\theta = x - \omega t$. Then Kodama reduction

$$a_0 + a_0 a_x + E = 0, \ E_x = \delta + a^N, \ a_t + \frac{1}{2} \left( \sum_{m=0}^{k} a^m a^{k-m} \right)_x = 0, \ k = 1, ..., N$$

becomes to the form of two ordinary differential equations of the first order ($u = a^0 - \omega$)

$$uu' + E = 0, \ E' = \delta + a^N,$$

while all other functions can be found iteratively

$$a^1 = -\frac{\gamma_1}{u}, \ a^2 = -\frac{1}{u} \left( \gamma_2 + \frac{1}{2} (a^1)^2 \right), ..., \ a^N = -\frac{1}{u} \left( \gamma_N + \frac{1}{2} \sum_{m=1}^{N-1} a^m a^{N-m} \right)$$

where $\gamma_k$ are integration constants.
Then periodic solution has the form ($\xi$ is an arbitrary constant)

$$u' = \delta + \frac{\xi}{u^2} + \frac{2}{u^2} \int u a^N(u) \, du$$

and can be integrated in hyperelliptic functions. If $N = 1$, then

$$u' = \delta - \frac{2 \gamma_1}{u} + \frac{\xi}{u^2};$$

if $N = 2$, then

$$u' = \delta - \frac{2 \gamma_2}{u} + \frac{\xi}{u^2} + \frac{\gamma_1^2}{u^3};$$

if $N = 3$, then

$$u' = \delta - \frac{2 \gamma_3}{u} + \frac{\xi}{u^2} + \frac{2 \gamma_1 \gamma_2}{u^3} + \frac{\gamma_1^3}{3u^5};$$

and so on.
Waterbag Reduction. Heaviside Step Function

Let us consider the ansatz

$$\lambda = \sum_{i=1}^{N-1} f_i \left[ \theta(p - a^i(t, x)) - \theta(p - a^{i+1}(t, x)) \right].$$

We assume that $a^i < a^{i+1}$, $f_i = \text{const} > 0$. Let us suppose that

$$f_i = -\sum_{n=1}^{i} \varepsilon_n \quad (i = 1, \ldots, N - 1), \quad f_N = \sum_{n=1}^{N} \varepsilon_n = 0.$$  

Then we have

$$\lambda = -\sum_{i=1}^{N} \varepsilon_i \theta(p - a^i(t, x)).$$

Substitution to the kinetic equation

$$\lambda_t + p\lambda_x - E\lambda_p = 0, \quad E_x = 1 - \int_{-\infty}^{\infty} \lambda dp, \quad E_t = \int_{-\infty}^{\infty} p\lambda dp.$$
Waterbag Reduction. Heaviside Step Function

yields $N$ component reduction

$$a_t^i + a_i a_x^i + E = 0, \quad (i = 1, \ldots, N)$$

$$E_x = 1 - \sum_{n=1}^{N-1} f_n(a^{n+1} - a^n) = 1 - \sum_{n=1}^{N} \epsilon_n a^n,$$

$$E_t = \frac{1}{2} \sum_{n=1}^{N-1} f_n [(a^{n+1})^2 - (a^n)^2] = \frac{1}{2} \sum_{n=1}^{N} \epsilon_n (a^n)^2,$$

which coincide with already found waterbag reduction via the moment decomposition

$$A^k = \frac{1}{k + 1} \sum_{n=1}^{N} \epsilon_n (a^n)^{k+1}, \quad \sum_{n=1}^{N} \epsilon_n = 0.$$
\[ \lambda = \sum_{i=1}^{N} b_i(t, x) \delta(p - a_i(t, x)) \]

Substitution to the kinetic equation

\[ \lambda_t + p\lambda_x - E\lambda_p = 0, \quad E_x = 1 - \int_{-\infty}^{\infty} \lambda dp, \quad E_t = \int_{-\infty}^{\infty} p\lambda dp \]

implies

\[ \sum_{i=1}^{N} \left[ (b_t^i + pb_x^i) \delta(p - a_i^i) - b_i^i (a_t^i + pa_x^i + E) \delta'(p - a_i^i) \right] = 0. \]

Taking into account the following equalities

\[ \varphi(p) \delta(p - a_i^i) = \varphi(a_i^i), \quad \varphi(p) \delta'(p - a_i^i) = -\varphi'(a_i^i), \]
we obtain

\[ a^i_t + a^i a^i_x + E = 0, \quad b^i_t + (a^i b^i)_x = 0, \quad (i = 1, ..., N) \]

\[ E_x = 1 - \sum_{n=1}^{N} b^n, \quad E_t = \sum_{n=1}^{N} a^n b^n. \]

This reduction is determined by the so called Zakharov moment decomposition

\[ A^k = \sum_{n=1}^{N} (a^n)^k b^n. \]
Generalizations

Let us consider more general ansatz

\[ \lambda = \sum_{i=1}^{N} \left[ c^i(t, x)\delta(p - a^i(t, x)) + e^i(t, x)\delta'(p - a^i(t, x)) \right]. \]

Substitution to the kinetic equation

\[ \lambda_t + p\lambda_x - E\lambda_p = 0, \quad E_x = 1 - \int_{-\infty}^{\infty} \lambda dp, \quad E_t = \int_{-\infty}^{\infty} p\lambda dp \]

implies

\[ \sum_{i=1}^{N} \left[ (c^i_t + pc^i_x)\delta(p - a^i) + (e^i_t - c^i a^i_t + p(e^i_x - c^i a^i_x) - Ec^i)\delta'(p - a^i) - e^i(a^i_t + pa^i_x + E)\delta''(p - a^i) \right] = 0. \]
Then we obtain

\[ c_t^i + (a^i c^i)_x - e_x^i = 0, \]

\[ c^i (a_t^i + a^i a_x^i + E) - (e_t^i + a^i e_x^i + 2e^i a_x^i) = 0, \]

\[ (a^i c^i - e^i)(a_t^i + a^i a_x^i + E) - a^i (e_t^i + a^i e_x^i + 2e^i a_x^i) = 0. \]

Introducing new functions \( b^i \) such that \( e^i = -(b^i)^2 / 2 \) we come back to the system

\[ a_t^i + a^i a_x^i + E = 0, \quad b_t^i + (a^i b^i)_x = 0, \quad c_t^i + \left( a^i c^i + \frac{(b^i)^2}{2} \right)_x = 0, \]

\[ E_x = 1 - \sum_{n=1}^{N} c^n, \quad E_t = \sum_{n=1}^{N} \left( a^n c^n + \frac{(b^n)^2}{2} \right) \]

is associated with the moment decomposition

\[ A^k = \sum_{n=1}^{N} \left[ (a^n)^k c^n + \frac{k}{2} (a^n)^{k-1} (b^n)^2 \right]. \]
Generalizations

Next $4N$ component nonlocal system

\begin{align*}
a_t^i + a^i a_x^i + E &= 0, \\
b_t^i + (a^i b^i)_x &= 0, \\
c_t^i + \left(a^i c^i + \frac{(b^i)^2}{2}\right)_x &= 0, \\
g_t^i + (a^i g^i + b^i c^i)_x &= 0, \\
dE &= (1 - A^0) dx + A^1 dt
\end{align*}

is associated with the moment decomposition

\begin{align*}
A^k &= \sum_{n=1}^{N} (a^n)^k g^n - k \sum_{n=1}^{N} (a^n)^{k-1} b^n c^n + \frac{k(k-1)}{6} \sum_{n=1}^{N} (a^n)^{k-2} (b^n)^3.
\end{align*}
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