

Reductions of Kinetic Equations to Finite Component Systems

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Vlasov (Collisionless Boltzmann) Kinetic Equation

$$\lambda_t + p\lambda_x - \lambda_p u_x = 0,$$

where (λ is a distribution function, x is a space coordinate, t is a time variable, p is a momentum)

$$u = \int_{-\infty}^{\infty} \lambda dp.$$

Following the approach developed in plasma physics, one can introduce an infinite set of moments

$$A^k = \int_{-\infty}^{\infty} p^k \lambda dp.$$

The Benney hydrodynamic chain is

$$A_t^k + A_x^{k+1} + kA^{k-1}A_x^0 = 0, \quad k = 0, 1, 2, \dots$$

Kinetic Equation Describing High Frequency Waves in Electron Plasma

$$\lambda_t + p\lambda_x - E\lambda_p = 0,$$

where

$$E_x = 1 - \int_{-\infty}^{\infty} \lambda dp, \quad E_t = \int_{-\infty}^{\infty} p\lambda dp.$$

Introducing moments of the distribution function

$$A^k = \int_{-\infty}^{\infty} p^k \lambda dp,$$

the collisionless kinetic equation can be re-written as the nonlocal chain

$$A_t^k + A_x^{k+1} + kA^{k-1}E = 0, \quad k = 0, 1, 2, \dots,$$

where

$$dE = (1 - A^0)dx + A^1dt$$

Waterbag Reduction

$$A^k = \frac{1}{k+1} \sum_{n=1}^N \epsilon_n (a^n)^{k+1}, \quad \sum_{n=1}^N \epsilon_n = 0,$$

where ϵ_i are arbitrary constants.

Then Benney hydrodynamic chain reduces to the form

$$a_t^i + \left(\frac{(a^i)^2}{2} + \sum_{n=1}^N \epsilon_n a^n \right)_x = 0,$$

where N is an arbitrary positive integer.

Nonlocal chain reduces to the nonlocal N component system

$$a_t^i + a^i a_x^i + E = 0, \quad dE = (1 - A^0) dx + A^1 dt.$$

Local Representation

The nonlocal chain

$$A_t^k + A_x^{k+1} + kA^{k-1}E = 0, \quad k = 0, 1, 2, \dots,$$

where

$$dE = (1 - A^0)dx + A^1 dt$$

can be written in the evolution form

$$E_t = A^1, \quad A_t^1 + A_x^2 + E(1 - E_x) = 0,$$

$$A_t^k + A_x^{k+1} + kA^{k-1}E = 0, \quad k = 2, 3, \dots,$$

where $A^0 = 1 - E_x$. In such a form this non-hydrodynamic chain has just one local conservation law of Energy

$$(A^2 + E^2)_t + A_x^3 = 0.$$

Local Representation

The nonlocal system

$$a_t^k + a^k a_x^k + E = 0, \quad dE = (1 - A^0)dx + A^1 dt$$

transforms to N component evolution system ($k = 2, 3, \dots, N$)

$$a_t^k + a^k a_x^k + E = 0,$$

$$E_t = \frac{1}{2} \sum_{n=2}^N \epsilon_n (a^n)^2 + \frac{1}{2\epsilon_1} \left(E_x + \sum_{n=2}^N \epsilon_n a^n - 1 \right)^2,$$

where

$$a^1 = \frac{1}{\epsilon_1} \left(1 - \sum_{n=2}^N \epsilon_n a^n - E_x \right).$$

Hamiltonian Structure

The waterbag reduction can be written in the Hamiltonian form

$$E_t = - \sum_{n=2}^N \frac{\delta \mathbf{H}}{\delta a^n}, \quad a_t^i = \frac{1}{\epsilon_i} \partial_x \frac{\delta \mathbf{H}}{\delta a^i} + \frac{\delta \mathbf{H}}{\delta E}, \quad i = 2, 3, \dots, N,$$

which is determined by the local Hamiltonian

$$\mathbf{H} = -\frac{1}{2} \int \left[E^2 + \frac{1}{3} \sum_{n=2}^N \epsilon_n (a^n)^3 + \frac{1}{3\epsilon_1^2} \left(E_x + \sum_{n=2}^N \epsilon_n a^n - 1 \right)^3 \right] dx,$$

the momentum

$$\mathbf{P} = \int \left[\frac{1}{2} \sum_{n=2}^N \epsilon_n (a^n)^2 + \frac{1}{2\epsilon_1} \left(\sum_{n=2}^N \epsilon_n a^n \right)^2 + \frac{E_x}{\epsilon_1} \sum_{n=2}^N \epsilon_n a^n + \frac{E_x^2}{2\epsilon_1} \right] dx$$

and $N - 2$ parametric Casimir

$$\mathbf{Q} = \int \sum_{n=2}^N \tilde{\epsilon}_n a^n, \quad \sum_{n=2}^N \tilde{\epsilon}_n = 0.$$

Periodic Solutions

Each field variable a^k depend on a sole phase $\theta = x - \omega t$.
Then the nonlocal system

$$a_t^k + a^k a_x^k + E = 0, \quad dE = (1 - A^0)dx + A^1 dt$$

reduces to an ODE system (ζ and β_k are arbitrary constants)

$$u'^2 = \frac{\zeta}{u^2} - \delta + \frac{2u}{3}\epsilon_1 \pm \frac{2}{3u^2} \sum_{k=2}^N \epsilon_k (u^2 + \beta_k^2)^{3/2},$$

where

$$a^1 = u + \omega, \quad a^k = \omega \pm [u^2 + \beta_k^2]^{1/2}, \quad k = 2, 3, \dots, N,$$

This is N parametric periodic solution, whose characteristic velocity is determined by

$$\omega = -\frac{1}{2\delta} \sum_{k=2}^N \epsilon_k \beta_k^2,$$

which follows from comparison both above expressions for E' .

Periodic Solutions

Since $E = -uu'$, its explicit expression is

$$E = -\sqrt{\zeta - \delta u^2 + \frac{2u^3}{3}\epsilon_1 \pm \frac{2}{3} \sum_{k=2}^N \epsilon_k (u^2 + \beta_k^2)^{3/2}}.$$

Then a corresponding solution of the distribution function $\lambda(x, t, p)$ can be found due to the inverse formula

$$\lambda(x, t, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iqp} \sum_{k=0}^{\infty} \frac{(-iq)^k}{k!} A^k dq,$$

where

$$A^k = \frac{1}{k+1} \sum_{n=2}^N \epsilon_n [(\omega \pm [u^2 + \beta_n^2]^{1/2})^{k+1} - (u + \omega)^{k+1}].$$

Kodama Reduction

N component Kodama reductions

$$a_t^0 + a^0 a_x^0 + E = 0, \quad E_x = \delta + a^N, \quad a_t^k + \frac{1}{2} \left(\sum_{m=0}^k a^m a^{k-m} \right)_x = 0, \quad k = 1, \dots, N$$

possess the local Hamiltonian structure

$$a_t^0 = -\frac{\delta \mathbf{H}_{(N)}}{\delta E}, \quad E_t = \frac{\delta \mathbf{H}_{(N)}}{\delta a^0}, \quad a_t^k = \partial_x \frac{\delta \mathbf{H}_{(N)}}{\delta a^{N-k}}, \quad k = 1, 2, \dots, N-1,$$

where the Hamiltonian is given by

$$\mathbf{H}_{(N)} = \frac{1}{2} \int \left[E^2 + (a^0)^2 (\delta - E_x) - \frac{a^0}{3} \sum_{k=1}^{N-1} a^k a^{N-k} - \frac{1}{3} \sum_{n=1}^{N-1} \sum_{k=0}^n a^k a^{n-k} a^{N-n} \right] dx$$

and the Momentum is

$$\mathbf{P} = \int \left[\frac{1}{2} \sum_{k=1}^{N-1} a^k a^{N-k} + a^0 E_x \right] dx.$$

Periodic Solutions

Each field variable a^i depend on a sole phase $\theta = x - \omega t$. Then Kodama reduction

$$a_t^0 + a^0 a_x^0 + E = 0, \quad E_x = \delta + a^N, \quad a_t^k + \frac{1}{2} \left(\sum_{m=0}^k a^m a^{k-m} \right)_x = 0, \quad k = 1, \dots, N$$

becomes to the form of two ordinary differential equations of the first order ($u = a^0 - \omega$)

$$uu' + E = 0, \quad E' = \delta + a^N,$$

while all other functions can be found iteratively

$$a^1 = -\frac{\gamma_1}{u}, \quad a^2 = -\frac{1}{u} \left(\gamma_2 + \frac{1}{2} (a^1)^2 \right), \dots, \quad a^N = -\frac{1}{u} \left(\gamma_N + \frac{1}{2} \sum_{m=1}^{N-1} a^m a^{N-m} \right)$$

where γ_k are integration constants.

Periodic Solutions

Then periodic solution has the form (ξ is an arbitrary constant)

$$u'^2 = \delta + \frac{\xi}{u^2} + \frac{2}{u^2} \int ua^N(u) du$$

and can be integrated in hyperelliptic functions. If $N = 1$, then

$$u'^2 = \delta - \frac{2\gamma_1}{u} + \frac{\xi}{u^2};$$

if $N = 2$, then

$$u'^2 = \delta - \frac{2\gamma_2}{u} + \frac{\xi}{u^2} + \frac{\gamma_1^2}{u^3};$$

if $N = 3$, then

$$u'^2 = \delta - \frac{2\gamma_3}{u} + \frac{\xi}{u^2} + \frac{2\gamma_1\gamma_2}{u^3} + \frac{\gamma_1^3}{3u^5};$$

and so on.

Waterbag Reduction. Heaviside Step Function

Let us consider the ansatz

$$\lambda = \sum_{i=1}^{N-1} f_i [\theta(p - a^i(t, x)) - \theta(p - a^{i+1}(t, x))].$$

We assume that $a^i < a^{i+1}$, $f_i = \text{const} > 0$. Let us suppose that

$$f_i = - \sum_{n=1}^i \varepsilon_n \quad (i = 1, \dots, N-1), \quad f_N = \sum_{n=1}^N \varepsilon_n = 0.$$

Then we have

$$\lambda = - \sum_{i=1}^N \varepsilon_i \theta(p - a^i(t, x)).$$

Substitution to the kinetic equation

$$\lambda_t + p\lambda_x - E\lambda_p = 0, \quad E_x = 1 - \int_{-\infty}^{\infty} \lambda dp, \quad E_t = \int_{-\infty}^{\infty} p\lambda dp$$

Waterbag Reduction. Heaviside Step Function

yields N component reduction

$$a_t^i + a^i a_x^i + E = 0, \quad (i = 1, \dots, N)$$

$$E_x = 1 - \sum_{n=1}^{N-1} f_n (a^{n+1} - a^n) = 1 - \sum_{n=1}^N \varepsilon_n a^n,$$

$$E_t = \frac{1}{2} \sum_{n=1}^{N-1} f_n [(a^{n+1})^2 - (a^n)^2] = \frac{1}{2} \sum_{n=1}^N \varepsilon_n (a^n)^2,$$

which coincide with already found waterbag reduction via the moment decomposition

$$A^k = \frac{1}{k+1} \sum_{n=1}^N \varepsilon_n (a^n)^{k+1}, \quad \sum_{n=1}^N \varepsilon_n = 0.$$

$$\lambda = \sum_{i=1}^N b_i(t, x) \delta(p - a^i(t, x))$$

Substitution to the kinetic equation

$$\lambda_t + p\lambda_x - E\lambda_p = 0, \quad E_x = 1 - \int_{-\infty}^{\infty} \lambda dp, \quad E_t = \int_{-\infty}^{\infty} p\lambda dp$$

implies

$$\sum_{i=1}^N [(b_t^i + pb_x^i)\delta(p - a^i) - b^i(a_t^i + pa_x^i + E)\delta'(p - a^i)] = 0.$$

Taking into account the following equalities

$$\varphi(p)\delta(p - a^i) = \varphi(a^i), \quad \varphi(p)\delta'(p - a^i) = -\varphi'(a^i),$$

we obtain

$$a_t^i + a^i a_x^i + E = 0, \quad b_t^i + (a^i b^i)_x = 0, \quad (i = 1, \dots, N)$$

$$E_x = 1 - \sum_{n=1}^N b^n, \quad E_t = \sum_{n=1}^N a^n b^n.$$

This reduction is determined by the so called Zakharov moment decomposition

$$A^k = \sum_{n=1}^N (a^n)^k b^n.$$

Generalizations

Let us consider more general ansatz

$$\lambda = \sum_{i=1}^N [c^i(t, x)\delta(p - a^i(t, x)) + e^i(t, x)\delta'(p - a^i(t, x))].$$

Substitution to the kinetic equation

$$\lambda_t + p\lambda_x - E\lambda_p = 0, \quad E_x = 1 - \int_{-\infty}^{\infty} \lambda dp, \quad E_t = \int_{-\infty}^{\infty} p\lambda dp$$

implies

$$\sum_{i=1}^N [(c_t^i + pc_x^i)\delta(p - a^i) + (e_t^i - c^i a_t^i + p(e_x^i - c^i a_x^i) - Ec^i)\delta'(p - a^i) - e^i(a_t^i + pa_x^i + E)\delta''(p - a^i)] = 0.$$

Then we obtain

$$c_t^i + (a^i c^i)_x - e_x^i = 0,$$

$$c^i (a_t^i + a^i a_x^i + E) - (e_t^i + a^i e_x^i + 2e^i a_x^i) = 0,$$

$$(a^i c^i - e^i)(a_t^i + a^i a_x^i + E) - a^i (e_t^i + a^i e_x^i + 2e^i a_x^i) = 0.$$

Introducing new functions b^i such that $e^i = -(b^i)^2/2$ we come back to the system

$$a_t^i + a^i a_x^i + E = 0, \quad b_t^i + (a^i b^i)_x = 0, \quad c_t^i + \left(a^i c^i + \frac{(b^i)^2}{2} \right)_x = 0,$$

$$E_x = 1 - \sum_{n=1}^N c^n, \quad E_t = \sum_{n=1}^N \left(a^n c^n + \frac{(b^n)^2}{2} \right)$$

is associated with the moment decomposition

$$A^k = \sum_{n=1}^N \left[(a^n)^k c^n + \frac{k}{2} (a^n)^{k-1} (b^n)^2 \right].$$

Generalizations

Next $4N$ component nonlocal system





$$a_t^i + a^i a_x^i + E = 0, \quad b_t^i + (a^i b^i)_x = 0, \quad c_t^i + \left(a^i c^i + \frac{(b^i)^2}{2} \right)_x = 0,$$

$$g_t^i + (a^i g^i + b^i c^i)_x = 0, \quad dE = (1 - A^0)dx + A^1 dt$$

is associated with the moment decomposition

$$A^k = \sum_{n=1}^N (a^n)^k g^n - k \sum_{n=1}^N (a^n)^{k-1} b^n c^n + \frac{k(k-1)}{6} \sum_{n=1}^N (a^n)^{k-2} (b^n)^3.$$

References

-  *S.A. Akhmanov, Yu.E. Diakov, A.S. Chirkin,*
Introduction in statistical radiophysics and optics, Moscow. Nauka
1981.
-  *R.C. Davidson,*
Methods in nonlinear plasma theory. Vol. **37** in Pure and Applied
Physics. Academic Press. New York. 1972. *T.P. Coffey,*
Breaking of large amplitude plasma oscillations, Physics of Fluids,
1971, vol. 14, pp. 1402-1406.
-  *A.K. Khe, A.A. Chesnokov,*
Propagation of nonlinear perturbations in a quasineutral collisionless
plasma. Journal of Applied Mechanics and Technical Physics **52** No. 5
(2011) 677-688.
-  *V.P. Silin,*
Introduction to the kinetic theory of gases. Moscow, FIAN (in
Russian) (1998). first edition. Nauka (1971).