Nonlinear hyperbolic integrodifferential equations and their applications in hydrodynamics

Alexander Chesnokov

Lavrentyev Institute of Hydrodynamics SB RAS

chesnokov@hydro.nsc.ru

This work was done jointly with V. Yu. Liapidevskii and A. K. Khe

International Conference Geometrical Methods in Mathematical Physics
Moscow, Russia, December 12–17, 2011

Outline

To the memory of professor V. M. Teshukov
who was my guide in science and life

- Concept of hyperbolicity for integrodifferential equations
  - Definitions
  - Example and distinctive features
- Applications in hydrodynamics
  - Shallow water equations for vertical and horizontal shear flows
  - Kinetic equation for quasineutral collisionless plasma flows
- Conclusion

Generalized characteristics

Let us consider the class of equations of the form

$$\mathbf{U}_t + \mathbf{A}\langle \mathbf{U}_x \rangle = \mathbf{G}, \tag{1}$$

where $\mathbf{U}(t, x, \lambda) = (U_1, ..., U_m)^{\mathrm{t}}$ — unknown vector function; $t \in R^+$, $x \in R$, $\lambda \in [0, 1]$ — independent variables; $\mathbf{G}(t, x, \lambda, \mathbf{U}) = (G_1, ..., G_m)^{\mathrm{t}}$ — given vector function; $\mathbf{A} : B \to B$ — linear operator, acting on variable λ; B — Banach space of vector functions (we suppose that $\mathbf{U}(t, x, \cdot) \in B$).

The characteristics of system (1) is given by the equation $x'(t) = k^\alpha(t, x)$, where k^α is the eigenvalue of the problem

$$(\mathbf{F}^\alpha, (\mathbf{A} - k^\alpha I)\langle \varphi \rangle) = 0, \tag{2}$$

whose nontrivial solution with respect to the functional $\mathbf{F}^\alpha = (F_1^\alpha, ..., F_m^\alpha)$ is defined in the class of locally integrable or generalized functions. The functional $\mathbf{F}^\alpha$ acts on the variable λ; the variables t and x are treated as parameters, I is the identity map, and $\varphi = (\varphi_1, ..., \varphi_m)^{\mathrm{t}} \in B$ is a trial function. Applying the functional $\mathbf{F}^\alpha$ to Eq. (1), we obtain the relation on the characteristic:

$$(\mathbf{F}^\alpha, \mathbf{U}_t + k^\alpha \mathbf{U}_x) = (\mathbf{F}^\alpha, \mathbf{G}). \tag{3}$$

Definition. V. M. Teshukov, 1985. System (1) is generalized hyperbolic if all eigenvalues $k^{\alpha}(t,x)$ are real and the set of eigenfunctionals $\{\mathbf{F}^{\alpha}\}$ is complete (if $(\mathbf{F}^{\alpha},\varphi)=0$ for all α and $\varphi\in B$ then $\varphi=0$).

This definition is a natural generalization of the theory of hyperbolic systems, where $\mathbf{A}$ is an operator in a finite space. The generalization to multidimensional case $\mathbf{x}\in R^{n}$ is straightforward. To find the characteristics of the system

$$\mathbf{U}_t+\sum_{i=1}^{n}\mathbf{A}^{i}\langle\mathbf{U}_{x_i}\rangle=\mathbf{G} \qquad (1')$$

one has to solve the following eigenvalues problem

$$\left(\mathbf{F}^{\alpha},\left(\tau^{\alpha}I+\sum_{i=1}^{n}\xi_i\mathbf{A}^{i}\right)\langle\varphi\rangle\right)=0.$$

Here B — Banach space of vector functions; $\varphi(\lambda)\in B$ — trial function; $\mathbf{A}^{i}:B\to B$ — linear operators acting on λ; $\xi_i(t,\mathbf{x})$ — given functions; $\tau^{\alpha}(t,\mathbf{x})$ — sought-for eigenvalue, corresponding to the eigenfunctional $\mathbf{F}^{\alpha}$.

System $(1')$ is t-generalized hyperbolic if for all real ξ_i eigenvalues τ^{α} are real and the set of eigenfunctionals $\{\mathbf{F}^{\alpha}\}$ is complete.

Example. Consider the following integrodifferential system of equations

$$u_t + uu_x + s\int_0^1 H_x\,d\lambda = -s'\int_0^1 H\,d\lambda, \quad H_t + (uH)_x = 0, \qquad (4)$$

where $s(x) > 0$ is given function. This system belongs to the class (1) with

$$\mathbf{U} = \begin{pmatrix} u(t,x,\lambda) \\ H(t,x,\lambda) \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} u & s\int_0^1 \dots d\lambda \\ H & u \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} -s'\int_0^1 H\,d\lambda \\ 0 \end{pmatrix}$$

Eigenfunctionals and relations on the characteristics. Taking into account that the components of the vector function $\varphi = (\varphi_1, \varphi_2)^{\mathrm{t}}$ do not depend on each other, from Eq. (2) we obtain the equalities

$$\begin{aligned} &(F_1, (u-k)\varphi_1) + (F_2, H\varphi_1) = 0, \\ &s\int_0^1 \varphi_2\,d\lambda (F_1, 1) + (F_2, (u-k)\varphi_2) = 0. \end{aligned} \qquad (2')$$

If $k = k^i$ is a root of the characteristic equation

$$\chi(k) = 1 - s \int_0^1 \frac{H\, d\lambda}{(u-k)^2} = 0, \tag{5}$$

then problem $(2')$ has a nontrivial solution $\mathbf{F}^i = (F_1^i, F_2^i)$. The action of the functional on trial function $\psi = (\psi_1, \psi_2)$ is given by

$$(\mathbf{F}^i, \psi) = s \int_0^1 \frac{H \psi_1\, d\lambda}{(u-k^i)^2} - s \int_0^1 \frac{\psi_2\, d\lambda}{u - k^i}\,.$$

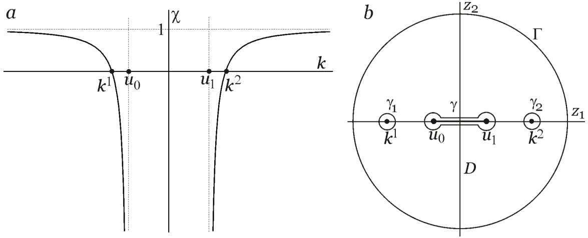
It what follows, we assume that $u_\lambda > 0$, $H > 0$. Characteristic equation (5) has two real roots: $k^1 < u_0 = u(t,x,0)$ and $k^2 > u_1 = u(t,x,1)$. Consider the segment $[u_0, u_1]$. Let $k = u(t,x,\nu)$. The functionals $\mathbf{F}^{1\nu}$ and $\mathbf{F}^{2\nu}$

$$\big(\mathbf{F}^{1\nu}, \psi(\lambda)\big) = -\psi_1'(\nu) + \frac{u_\nu}{H}\psi_2(\nu)\,;$$

$$(\mathbf{F}^{2\nu}, \psi) = \psi_1(\nu) + s \int_0^1 \frac{H(\lambda)(\psi_1(\lambda) - \psi_1(\nu))\, d\lambda}{(u(\lambda)-u(\nu))^2} - s \int_0^1 \frac{\psi_2(\lambda)\, d\lambda}{u(\lambda) - u(\nu)}$$

are a solution of Eqs. $(2')$.

The eigenvalues problem has discrete $\boldsymbol{k = k^i}$ and continuous $\boldsymbol{k = u}$ spectrum of characteristic velocities.



Let us introduce the functions

$$\boldsymbol{R = u - s\int\limits_0^1 \frac{H'\,d\lambda'}{u' - u}, \quad \omega = \frac{u_\lambda}{H}, \quad r^i = k^i - s\int\limits_0^1 \frac{H'\,d\lambda'}{u' - k^i}}\,.$$

Here $\boldsymbol{u = u(t,x,\lambda)}$, $\boldsymbol{u' = u(t,x,\lambda')}$, $\boldsymbol{H' = u(t,x,\lambda')}$. The characteristic relations (3) for the system (4) have the form

$$\boldsymbol{R_t + uR_x = Q(u), \quad \omega_t + u\omega_x = 0, \quad r^i_t + k^i r^i_x = Q(k^i)},$$

where

$$Q(z) = -s'(x) \int_0^1 \frac{u'H'\,d\lambda'}{u' - z}\,.$$

If $s = \mathbf{const}$, then the quantities R, ω, and r^i are Riemann invariants. Hyperbolicity conditions are formulated in terms of the analytical function $\chi(z)$, or more precisely, its limiting values on the segment $[u_0, u_1]$:

$$\chi^{\pm}(u) = 1 + \frac{s}{\omega_1(u_1 - u)} - \frac{s}{\omega_0(u_0 - u)} - s\int_0^1 \left(\frac{1}{\omega'}\right)_{\lambda'} \frac{d\lambda'}{u' - u} \mp \frac{g\pi i}{u_\lambda}\left(\frac{1}{\omega}\right)_\lambda$$

Theorem. The conditions

$$\chi^{\pm} \neq 0, \quad \Delta \arg \frac{\chi^{+}(u)}{\chi^{-}(u)} = 0 \tag{6}$$

($\Delta \arg \chi^{\pm}$ is the increment of the argument of the complex function $\chi^{\pm}$ as λ changes from 0 to 1 for fixed t and x) are necessary and sufficient for hyperbolicity of Eqs. (4) if the functions u, $H > 0$, and ω are differentiable and the functions $u_\lambda > 0$ and ω_λ satisfy the Holder condition over the variable λ.

Distinctive features. The main distinction between finite-dimensional and infinite-dimensional cases consist in appearance of the continuous spectra of characteristic velocities. If the values of characteristic velocities belong to a bounded set, then (1) describes processes with finite speed of propagation of perturbations in the x-direction.

The basic results in this field were obtained by V. M. Teshukov (generalized hyperbolicity, local solvability of the Cauchy problem, exact solutions,...).

Development of the theory of integrodifferential equations includes

- Derivation of new physical relevant integrodifferential models;
- Formulation of hyperbolicity conditions and stability analysis;
- Proof of the existence and uniqueness of the Cauchy problem;
- Construction of exact solutions and their physical interpretation;
- Theory of discontinuous solutions;
- Conservation laws and numerical simulations...

Shallow water equations for shear flows

We consider the motion of an ideal incompressible fluid with the free boundary $z = h(t, x, y)$ in an open channel with even bottom $z = 0$ in a gravity field

$$u_t + (\mathrm{v}\cdot\nabla)u + p_x = 0, \quad \varepsilon^2(v_t + (\mathrm{v}\cdot\nabla)v) + p_y = 0,$$
$$\varepsilon^2(w_t + (\mathrm{v}\cdot\nabla)w) + p_z = -g, \quad \nabla\cdot\mathrm{v} = 0;$$
$$h_t + uh_x + vh_y - w = 0, \quad p = p_0 \quad (z = h);$$
$$w = 0 \quad (z = 0).$$

Here $\varepsilon = l/L$ is a dimensionless small parameter; the quantity L specifies the characteristic scale on the x axis directed along the channel, and the quantity l on the y and z axes.
In the long wave approximation ($\varepsilon \to 0$) this model takes a form

$$u_t + uu_x + vu_y + wu_z + gh_x = 0, \quad h_y = 0,$$
$$w = -\int\limits_0^z (u_x + v_y)\,dz, \quad h_t + \Big(\int\limits_0^h u\,dz\Big)_x + \Big(\int\limits_0^h v\,dz\Big)_y = 0.$$

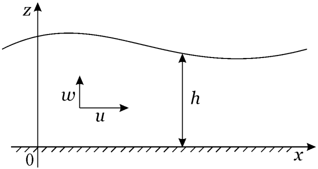
Vertical shear flows. Under assumptions $u_y = v_y = 0$, one obtains D. Benney equations for the unknown functions $u(t,x,z)$ and $h(t,x)$:

$$u_t + uu_x + wu_z + gh_x = 0, \quad h_t + \Big(\int_0^h u\,dz \Big)_x = 0, \quad w = -\int_0^z u_x\,dz.$$

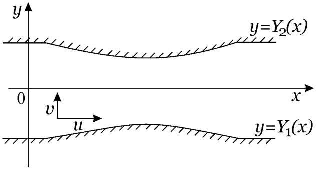
Horizontal shear flows. Consider the class of fluid flows in which $u_z = 0$, $v_z = 0$. The long-wave model becomes (Chesnokov, Liapidevskii, 2009):

$$u_t + uu_x + vu_y + gh_x = 0, \quad h_y = 0,$$

$$h_t + (uh)_x + (vh)_y = 0, \quad uY_i'(x) - v\big|_{y=Y_i} = 0.$$



Plane-parallel vertical shear fluid flow in an open channel with straight lateral walls: section by the plane $y = \mathrm{const}$.



Horizontal shear fluid flow in an open channel with curve lateral walls: section by the plane $z = \mathrm{const}$.

It is convenient to transform to semi-Lagrangian coordinates by the change of the variable $z = \Phi(t,x,\lambda)$ (or $y = \Phi$), where the function Φ is a solution of the Cauchy problem (V. E. Zakharov, 1980)

$$\Phi_t + u(t,x,\Phi)\Phi_x = \begin{cases} w(t,x,\Phi) \\ v(t,x,\Phi) \end{cases} \qquad \Phi\Big|_{t=0} = \begin{cases} \lambda h(0,x) \\ \lambda Y_2(x) + (1-\lambda) Y_1(x) \end{cases}$$

In the new variables, the functions

$$u(t,x,\lambda), \quad H(t,x,\lambda) = \begin{cases} \Phi_\lambda \\ h(t,x)\Phi_\lambda \end{cases}$$

are described by the integrodifferential system of equations

$$u_t + u u_x + \left(s \int_0^1 H \, d\lambda \right)_x = 0, \quad H_t + (uH)_x = 0; \quad s(x) = \begin{cases} g \\ g/Y \end{cases}$$

Here $Y(x) = Y_2(x) - Y_1(x)$ is the width of the channel.

Thus, we arrive to the integrodifferential system (4), which is generalized hyperbolic under conditions (6).

Sub- and supercritical steady-state horizontal-shear flows. We take the stream function as the Lagrangian coordinate λ, so that

$$H = hy_\lambda = h/\lambda_y = 1/u$$

To be specific, let $u > 0$. Then, as a result of integrating Eqs. (4) we obtain

$$u = \sqrt{2(C(\lambda) - gh)}, \quad H = 1/\sqrt{2(C(\lambda) - gh)}$$

Here $C(\lambda) > 0$ is an arbitrary function and the fluid depth $h(x)$ can be found from the closing relation

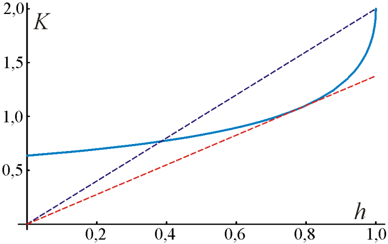
$$K(h) = Y(x)h, \quad K(h) = \int_0^1 \frac{d\lambda}{\sqrt{2(C(\lambda) - gh)}}$$

The steady-state flow for which the following inequality holds

$$S = 1 - \frac{K'(h)}{Y(x)} = 1 - \frac{g}{Y}\int_0^1 \frac{H\,d\lambda}{u^2} < 0$$

will be called subcritical and the flow for which the inverse inequality holds will be called supercritical.

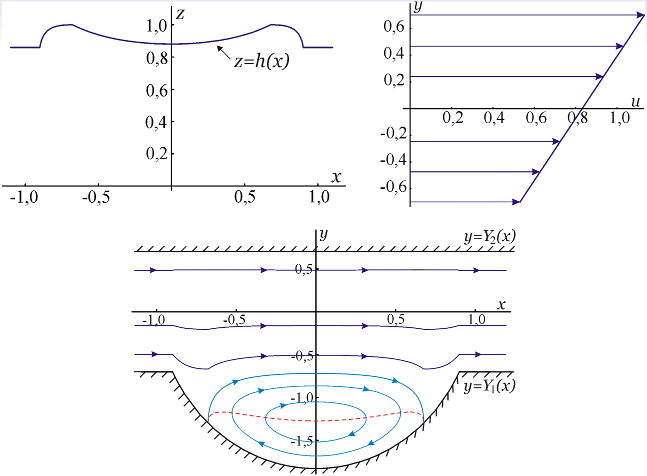
In view of $\boldsymbol{h'(x) = -(YS)^{-1}Y'(x)h}$
in the subcritical flow regime the fluid depth $\boldsymbol{h(x)}$ increases (decreases) with increase (decrease) in the channel cross-section $\boldsymbol{Y(x)}$, while in the supercritical regime $\boldsymbol{h(x)}$ decreases (increases) with increase (decrease) in $\boldsymbol{Y(x)}$.



Exact solutions describing different flow regimes were constructed and their properties were studied (Chesnokov, Liapidevskii, 2009).

Flow with recirculation zones. Subcritical flow past a local channel expansion: the continuation of solution with the inclusion of the recirculation region

$$\boldsymbol{u = \mp\sqrt{2(G(\lambda) - gh)}, \quad H = \mp 1/\sqrt{2(G(\lambda) - gh)}}$$



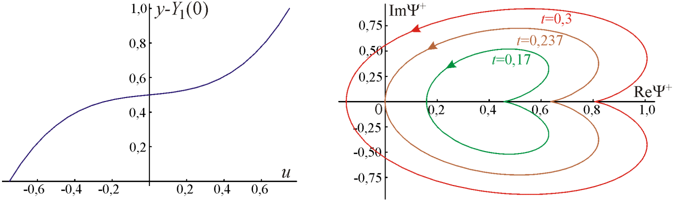
Fluid depth, velocity profile (before entering the zone of the channel expansion) and streamlines in a subcritical steady-state flow with a recirculation zone.

Similar solutions are known for the vertical-shear flows over an uneven bottom (E. Varley, P. Blythe, 1983; V. Teshukov and A. Budlal, 2006).

Validation of the hyperbolicity conditions. We consider the exact solution of Eqs. (4)

$$\boldsymbol{u = \big(x + C(\lambda)\big)t^{-1}, \quad H = t^{-1}; \quad C^3 + aC - \lambda + 1/2 = 0}$$

that describes the fluid spreading under the pressure action in an open channel of constant cross-section. Let us verify the hyperbolicity conditions (6) for the solution considered.



Velocity profile $\boldsymbol{u}$ for fixed $\boldsymbol{t}$ and $\boldsymbol{x}$; parametric representation of the real and imaginary parts of the function $\boldsymbol{\Psi^{\pm}}$ are presented. Here $\boldsymbol{g = 1}$, $\boldsymbol{Y = 1}$, $\boldsymbol{\Psi^{\pm} = (u_1 - u)(u - u_0)\chi^{\pm}(u)}$; $\boldsymbol{a = 5/48}$, $\boldsymbol{u_1 = -u_0 = 3/4}$.

This example shows that the system (4) can change its type in the process of flow evolution, which corresponds to long-wave instability for the considered velocity distribution.

Hyperbolicity and the classical stability criteria. Shallow water equations for vertical-shear flows under lid

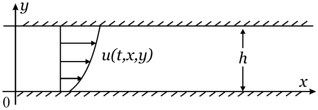
$$u_t + uu_x + vu_y + \rho^{-1}p_x = 0,\ p_y = -\rho g,\ u_x + v_y = 0,\ v\big|_{y=0;h} = 0$$

This model is hyperbolic for flows with monotonic velocity profile if conditions (6) are satisfied with function

$$\chi(z) = \int_{u_0}^{u_1} \frac{dy}{(u-z)^2}$$

(Chesnokov, 1998). Consider the class of solutions

$$u = U(y), \quad v = 0, \quad p = \rho g(h-y) + p_0(t)$$



Proposition (Chesnokov, Knyazeva, 2011) Let smooth and monotonic velocity profile $U(y)$ has no more than one point of inflection. Then satisfying to one of the stability criterion (Rayleigh, Fjortoft, Rosenbluth — Simon) is a necessary and sufficient condition for hyperbolicity of this model.

Vlasov-like formulation: special class of solutions. Consider the vertical shear flow (horizontal shear flow in channel of constant cross-section) with nonzero vorticity $\omega = u_y$ (potential vorticity $\omega = u_y/h$). Shallow water equations shear for flows admit the kinetic formulation:

$$W_t + uW_x - gh_xW_u = 0, \quad h = \int_{u_0}^{u_1} W\,du,$$

$$u_{1t} + u_1u_{1x} + gh_x = 0, \quad u_{0t} + u_0u_{0x} + gh_x = 0.$$

Here $W(t, x, u) = \omega^{-1}$ (the analog of the distribution function) is considered as an unknown function depending on t, x, u; functions $u_0(t, x)$ and $u_0(t, x)$ are velocities at the bottom and at free surface (or at lateral walls).
Let us introduce the new function

$$R(t, x, u) = u - g\int_{u_0}^{u_1} \frac{W(t, x, u')\,du'}{u' - u},$$

which satisfies the same equation satisfied by W, namely

$$R_t + uR_x - gh_xR_u = 0.$$

Consider the solutions with functionally dependent invariants W and R:

$$W = \phi(R) = \phi\Big(u - g\int_{u_0}^{u_1} \frac{W(t,x,u')\,du'}{u'-u}\Big)$$

To construct the solution is necessary:

1) to define the function $W = \bar{W}(u_0, u_1, u)$ from the integral equation;

2) to solve closed system of differential equations

$$u_{1t} + u_1 u_{1x} + g\big(\bar{h}(u_0,u_1)\big)_x = 0,$$

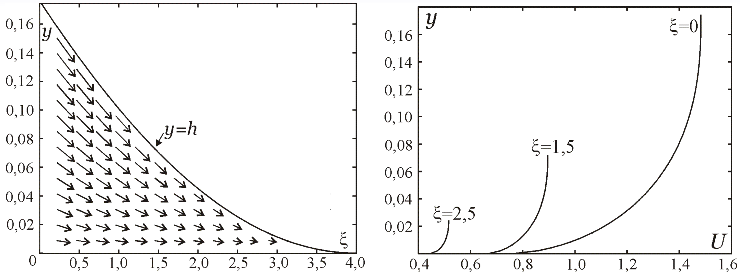
$$u_{0t} + u_0 u_{0x} + g\big(\bar{h}(u_0,u_1)\big)_x = 0; \quad \bar{h}(u_0,u_1) = \int_{u_0}^{u_1} \bar{W}(u_0,u_1,u)\,du.$$

Let $\phi = a(R-b)$, $a^{-1} = g\pi\cot(\mu\pi)$, $(0 < \mu < 1,\ \mu \neq 1/2)$. Then the solution of the integral equation of the form

$$\bar{W} = \frac{\sin(\mu\pi)}{g\pi}\Big(u - b - \mu(u_1 - u_0)\Big)\Big(\frac{u-u_0}{u_1-u}\Big)^{\mu}$$

Teshukov, Russo, Chesnokov, 2004; 2005; Chesnokov, Kovtunenko, 2011

Example. Simple wave: $W = \bar{W}(\xi, u)$, $u_i = u_i(\xi)$, $\xi = x/t$.



Free surface $y = h(\xi)$ and relative velocity profile $U = u(\xi, y) - \xi$.
The system of conservation laws

$$h_t + m_x = 0, \quad m_t + \left(n + gh^2/2\right)_x = 0;$$

$$m = \int_{u_0}^{u_1} uW\,du, \quad n = \int_{u_0}^{u_1} u^2 W\,du$$

is used in the description of discontinuous solutions from special class.

In the study of shear discontinuous flows we propose to use the conservation laws (in semi-Lagrangian variables)

$$u_t + uu_x - u_{0t} - u_0 u_{0x} = 0, \quad H_t + (uH)_x = 0,$$

$$\Big(\int_0^1 uH d\lambda \Big)_t + \Big(\int_0^1 u^2 H d\lambda + \frac{g}{2} \Big(\int_0^1 H d\lambda \Big)^2 \Big)_x = 0,$$

which express the conservation of the relative local momentum, mass and total momentum of the liquid layer.

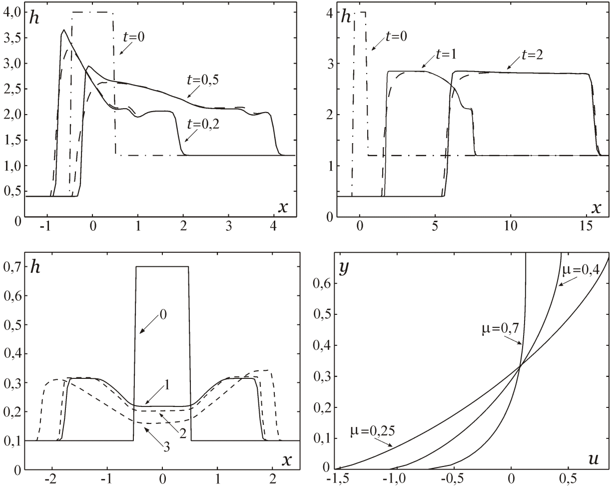
By applying piecewise linear approximation for velocity profile, we obtain the following system of differential conservation laws

$$\frac{\partial h_i}{\partial t} + \frac{\partial}{\partial x}(u_{ci} h_i) = 0, \quad \frac{\partial}{\partial t}(\omega_i h_i) + \frac{\partial}{\partial x}(u_{ci}\omega_i h_i) = 0,$$

$$\frac{\partial m}{\partial t} + \frac{\partial}{\partial x}\left(\sum_{i=1}^{N} \left(u_{ci}^2 h_i + \frac{\omega_i^2 h_i^3}{12} \right) + \frac{g}{2} \Big(\sum_{i=1}^{N} h_i \Big)^2 \right) = 0$$

(averaged velocities in the layers u_{ci} are computed with using h_i, ω_i and m).

Numerical results, obtained for discontinuous flows: transition to the self-similar regime (comparison the full model and a special class of solutions), the influence of the shear (comparison between the classical model and model for shear flow). The calculations are performed using Nessyahu–Tadmor (1990) scheme.



Hugoniot conditions: full model and special class of solutions. Proposed conservation laws for shear flows give the following Hugoniot conditions at the shock front $x = x(t)$ moving with the velocity $V = x'(t)$

$$\left[(u - V)^2 - (u_0 - V)^2\right] = 0, \quad \left[H(u - V)\right] = 0,$$

$$\left[\int_0^1 (u - V)^2 H\, d\lambda + \frac{g}{2}\left(\int_0^1 H\, d\lambda\right)^2\right] = 0$$

From these equations it follows that $[W] = 0$.

The dependence $[R - b - g\pi W \cot(\mu\pi)] = 0$ is not fulfilled in the general case. However, for flows with a weak discontinuity it is acceptable. In fact, if $\delta = [h]$ then $[R] = O(\delta^2)$. Consequently, with accuracy $O(\delta^2)$ we can use following shock conditions

$$\left[R - b - g\pi W \mathrm{ctg}(\mu\pi)\right] = 0, \quad \left[H(u - V)\right] = 0,$$

$$\left[\int_0^1 (u - V)^2 H\, d\lambda + \frac{g}{2}\left(\int_0^1 H\, d\lambda\right)^2\right] = 0$$

and perform calculations in frame of special class of solutions.

Kinetic model for quasineutral collisionless plasma flows

The equation of motion of a quasineutral collisionless plasma has the form (Gurevich, Pitaevskii, 1980)

$$\frac{\partial f^1}{\partial t} + u\frac{\partial f^1}{\partial x} - \frac{e}{M_i}\frac{\partial \varphi}{\partial x}\frac{\partial f^1}{\partial u} = 0, \quad \varphi = \frac{T_e}{e}\ln\left(\frac{1}{N_0}\int\limits_{-\infty}^{\infty} f^1\,du\right)$$

Here $f^1(t,x,u)$ is the ion distribution function, x and t are the spatial coordinate and time, $\varphi(t,x)$ is the electric field potential, u and M_i are the velocity and mass of the ions, e and T_e are the charge and temperature of electrons, and N_0 is the density of the unperturbed plasma.

Consider solutions in the class of functions that are piecewise continuous functions in the variable u with a bounded support:

$$f^1(t,x,u) = f(t,x,u)\Big[\theta\big(u - v_0(t,x)\big) - \theta\big(u - v_1(t,x)\big)\Big].$$

Here θ is a Heaviside function, v_0 and v_1 are the boundaries of the interval in the variable u, beyond which the distribution function f^1 is identically zero.

Substitution of the representation of solution into the kinetic equation yields

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} - \frac{b}{n}\frac{\partial n}{\partial x}\frac{\partial f}{\partial u} = 0, \quad n = \int_{v_0}^{v_1} f\,du, \quad \left(b = \frac{T_e}{M_i}\right)$$
$$\frac{\partial v_0}{\partial t} + v_0\frac{\partial v_0}{\partial x} + \frac{b}{n}\frac{\partial n}{\partial x} = 0, \quad \frac{\partial v_1}{\partial t} + v_1\frac{\partial v_1}{\partial x} + \frac{b}{n}\frac{\partial n}{\partial x} = 0 \qquad (7)$$

To investigate the properties of the kinetic model (7), it is appropriate to transform to the semi-Lagrangian variables (x, λ) by the change of variable $u = u(t, x, \lambda)$, where $u(t, x, \lambda)$ is a solution of the Cauchy problem

$$u_t + uu_x = -bn^{-1}n_x, \quad u\big|_{t=0} = u^0(x, \lambda) \quad \big(\lambda \in [0, 1]\big)$$

For the new desired functions $u(t, x, \lambda)$ and $H(t, x, \lambda) = u_\lambda f(t, x, \lambda) > 0$, we obtain the system of integrodifferential equations (Chesnokov, Khe, 2011)

$$u_t + uu_x + \frac{b}{n}\int_0^1 H_x\,d\lambda = 0, \quad H_t + (uH)_x = 0, \quad n = \int_0^1 H\,d\lambda \quad (7')$$

This system belongs to the class of equations (1).

Generalized hyperbolicity. Characteristic function has the form

$$\chi(k) = 1 - \frac{b}{n} \int_0^1 \frac{H\, d\lambda}{(u-k)^2} = 0 \tag{5'}$$

Spectral problem has discrete $k = k^1 < v_0(t,x)$, $k = k^2 < v_1(t,x)$ and continuous $k = k^\nu = u(t,x,\nu)$ characteristic spectrum.

The conditions

$$\chi^+(u) \neq 0, \quad \Delta \arg \frac{\chi^+(u)}{\chi^-(u)} = 0 \tag{6'}$$

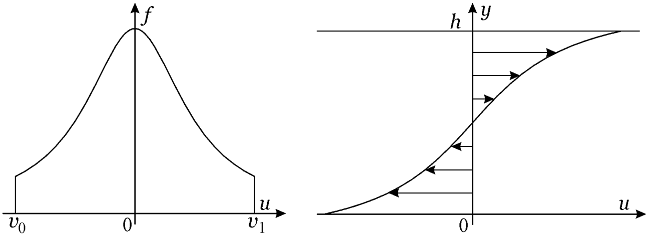
are necessary and sufficient for hyperbolicity of Eqs. $(7')$ if the functions $u(t,x,\lambda)$ and $f(t,x,\lambda) > 0$ are differentiable and the functions $u\lambda > 0$ and $f_\lambda(t,x,\lambda)$ satisfy the Holder condition with respect to the variable $\lambda \in [0,1]$.

We note another possible interpretation of the model (7). The model

$$u_t + uu_x + vu_y + p_x = 0, \quad p = p(h)$$

$$h_t + \Big(\int_0^h u\, dy \Big)_x = 0, \quad v = - \int_0^y u_x\, dy$$

describes plane-parallel shear flow of an ideal fluid in a long channel with an elastic wall (Chesnokov, 2001). If $p(h) = b \ln h + p_0$ then the model is converted into the kinetic equations (7) for plasma flows by introducing the new dependent variables $f = 1/u_y$, $v_0 = u(t, x, 0)$, $v_1 = u(t, x, h)$ and the independent variables t, x, u.



Dispersion relations similar to Eq. $(5')$ occur in studies of the propagation of small perturbations in a plasma. It is known that solutions described by distribution functions with one maximum are stable. These functions obey the inequality $(u_c - u)f'(u) \geq 0$, where $u = u_c$ is an extreme point. For shear flows ($u'(y) = 1/f(u)$) this means that $u''(y)(u(y) - u_c) \geq 0$, which is Fjortoft stability criterion.

Conditions $(6')$ allow one to verify whether the equations of motion $(7')$ are hyperbolic for a given solution u, H. In the plane (Z_1, Z_2) we construct a closed contour C, which is given by the equations

$$Z^1 = m(u)\mathrm{Re}\{\chi^{\pm}(u)\}, \quad Z^2 = \pm m(u)\mathrm{Im}\{\chi^{\pm}(u)\}$$

Here $m(u) = (v_1 - u)(u - v_0)$. If the point of $Z^1 = 0$, $Z^2 = 0$ lies in the domain bounded by the contour C, then the equation $(5')$ has complex roots. Otherwise, the equations of motion for the solution are hyperbolic.

The hyperbolicity conditions agree with the known stability criteria: conditions $(6')$ are satisfied for the distribution functions with a single maximum. Since

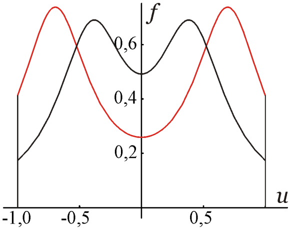
$$Z^1(v_0) = (v_1 - v_0)f_0 > 0, \quad Z^1(v_1) = (v_1 - v_0)f_1 > 0,$$

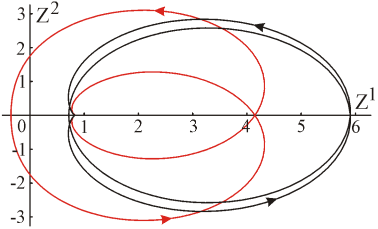
conditions $(6')$ may be violated if $Z^1(u_*) < 0$, $Z^2(u_*) = 0$ at some interior point $u_* \in (v_0, v_1)$. The equality $Z^2(u) = 0$ is satisfied at the single interior point of $u = u_c$. At this point

$$Z^1(u_c) = m_c + \frac{b}{n}\left((u_c - v_0)f_1 + (v_1 - u_c)f_0 + m_c \int\limits_{v_0}^{v_1} \frac{(u_c - v)f'\,dv}{(u_c - v)^2}\right) > 0$$

For the distribution functions $f(u)$ with one local maximum, the hyperbolicity conditions $(6')$ cannot be violated.

Let us verify the hyperbolicity conditions for the distribution function with two maxima. Graphs show two distribution functions (black and red lines) and the corresponding them contours C^+.





It is evident that the hyperbolicity conditions $(\mathbf{6'})$ are satisfied for the distribution function whose maxima are closely spaced (the black curve).

Instability is possible if the maxima are far enough from each other and have a relatively large amplitude.

Indeed, for the distribution shown in figure by red curve, conditions $(\mathbf{6'})$ are violated. The argument of the functions $\chi^+(u)$ gains an increment 2π (the functions $-\chi^-(u)$ give the same increment). Thus, the characteristic equation $(\mathbf{5'})$ has a complex root and a complex conjugate root.

Traveling waves. The solution of Eqs. (7) of the form $f = f(\zeta, u)$, $v_i = v_i(\zeta)$, $\zeta = x - Dt$ describes a traveling wave. System (7) becomes

$$(u - D)f_\zeta - \varphi_\zeta f_u = 0, \quad \varphi(\zeta) = b \ln \Big(\int_{v_0}^{v_1} f \, du \Big),$$

$$(v_j - D)v_{j\zeta} + \varphi_\zeta = 0 \qquad (j = 0, 1)$$

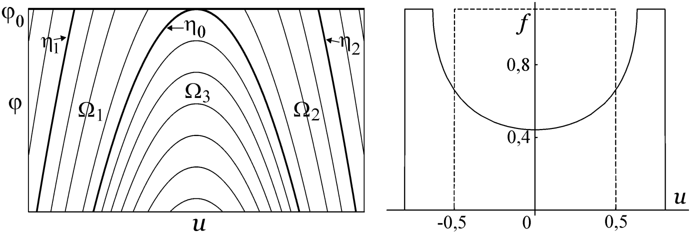
Integration yields (analog of Bernstein-Green-Kruskal waves, but in this case $\Psi(\eta)$ is not arbitrary function)

$$f = \Psi(\eta), \quad \eta = \frac{(u - D)^2}{2} + \varphi$$

Construct traveling wave which is continuously adjacent to a specified value $f = f_0(u)$ on the characteristic $\zeta = \zeta_0$. Instead of the variable ζ, it is expedient to use the variable φ; the dependence $\varphi = \varphi(\zeta)$ can be specified arbitrarily. Consider the Cauchy problem

$$f(\varphi_0, u) = f_0(u), \quad v_i(\varphi_0) = v_{0i} \qquad \Big(\exp(\varphi_0/b) = \int_{v_{00}}^{v_{10}} f_0(u) \, du \Big)$$

Let $v_{00} < D < v_{10}$ (flow with critical layer). Construct solution in domain $\varphi_m \le \varphi \le \varphi_0$. For $\varphi > \varphi_0$ the Cauchy problem is ill-posed.



Solution takes constant values on the characteristics $\eta = \mathbf{const}$. In the domains Ω_1, Ω_2 the solution is constructed by the method of characteristics:

$$\Psi_1(\eta) = f_0\big(D - \sqrt{2(\eta - \eta_0)}\big), \qquad \Psi_2(\eta) = f_0\big(D + \sqrt{2(\eta - \eta_0)}\big)$$

$$f(\varphi, u) = \Psi_1\big((u - D)^2/2 + \varphi\big), \qquad f(\varphi, u) = \Psi_2\big((u - D)^2/2 + \varphi\big)$$

To determine $f(\varphi, u) = \Psi_3(\eta)$ in Ω_3 we obtain the Abel equation

$$\int_{\varphi}^{\varphi_0} \frac{\Psi_3(\eta)\,d\eta}{\sqrt{\eta - \varphi}} = F(\varphi) = \frac{\exp(\varphi/b)}{\sqrt{2}} - \frac{1}{2}\Big(\int_{\varphi_0}^{\eta_1} \frac{\Psi_1(\eta)\,d\eta}{\sqrt{\eta - \varphi}} + \int_{\varphi_0}^{\eta_2} \frac{\Psi_2(\eta)\,d\eta}{\sqrt{\eta - \varphi}}\Big)$$

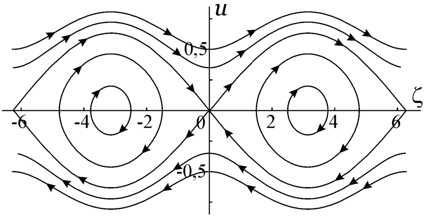
In view the equality $F(\varphi_0) = 0$, the solution of the Abel equation has the form

$$f(\varphi, u) = \Psi_3(\eta) = -\frac{1}{\pi} \int_{\eta}^{\varphi_0} \frac{F'(\varphi)\, d\varphi}{\sqrt{\varphi - \eta}}$$

Example: $f_0 = \mathbf{const}$; the plots were obtained for

$$b = 1;\ D = 0;\ v_{10} = -v_{00} = 1/2;\ \varphi_0 = -\varphi_m = 0{,}1;\ f_0 \approx 1{,}1$$

The dependence $\varphi = \varphi(\zeta)$ is arbitrary. Let $\varphi(\zeta) = \varphi_0 \cos \zeta$ then we have periodic solution.



Particle trajectories $d\zeta/dt = u - D$, $du/dt = -\varphi'(\zeta)$ in the phase plane (ζ, u) have the cat-eye shape characteristic of flows with a critical layer.

Approximate differential model. The evolution of the smooth solution of the generalized hyperbolic system of equations can involve a gradient catastrophe. The further description of the solutions is possible only for the class of discontinuous functions, which leads to the need to formulate the model $(7')$ in the form of conservation laws

$$u_{\lambda t} + (u u_\lambda)_x = 0, \quad H_t + (uH)_x = 0,$$

$$A_{1t} + (A_2 + bn)_x = 0, \quad A_l = \int_0^1 u^l H \, d\lambda \quad (l = 1, 2) \tag{8}$$

It is easy to show that (8) and $(7')$ are equivalent for smooth solutions.

To derive differential conservation laws that approximate the model (8), we perform a partition along the variable λ $(0 = \lambda_0 < \lambda_1 < ... < \lambda_M = 1)$ and use a piecewise constant approximation of the distribution function

$$f(t, x, u) = f_{ci}(t, x), \quad u \in [u_{i-1}, u_i]$$

with averaging of Eqs. (8) over λ. Here $u_i(t, x) = u(t, x, \lambda_i)$, $i = 1, ..., M$. Integrating (8) over the Lagrangian variable λ from λ_{i-1} to λ_i using that $H \, d\lambda = f \, du$, we obtain a system of conservation laws consisting of $2M + 1$

differential equations for the unknown functions $h_i(t,x)$, $f_{ci}(t,x)$, $A_1(t,x)$:

$$\frac{\partial h_i}{\partial t} + \frac{\partial}{\partial x}\Big(u_{ci}h_i\Big) = 0, \quad \frac{\partial}{\partial t}\Big(f_{ci}h_i\Big) + \frac{\partial}{\partial x}\Big(u_{ci}f_{ci}h_i\Big) = 0,$$
$$\frac{\partial A_1}{\partial t} + \frac{\partial}{\partial x}\left(\sum_{i=1}^{M}\left(u_{ci}^2 f_{ci}h_i + \frac{f_{ci}h_i^3}{12} + bf_{ci}h_i\right)\right) = 0 \tag{9}$$

The quantities $u_{ci}(t,x)$ are given by the formulas

$$u_{ci} = \frac{h_i}{2} + \sum_{k=1}^{i-1} h_k + \left(\sum_{i=1}^{M} f_{ci}h_i\right)^{-1}\left(A_1 - \sum_{i=1}^{M}\frac{f_{ci}h_i^2}{2} - \sum_{i=2}^{M} f_{ci}h_i\sum_{k=1}^{i-1}h_k\right)$$

In some cases, the system (9) proposed for the numerical simulation of plasma waves reduces to the known hydrodynamic limits of the kinetic equation (7). For $M=1$, system (9) reduces to the gas-dynamic type equations

$$(n/f)_t + (vn/f)_x = 0, \quad n_t + (vn)_x = 0,$$
$$(vn)_t + (v^2n + p)_x = 0; \quad p(n,f) = \frac{n^3}{12f^2} + bn,$$

where $n = h_1 f_{c1}$, $v = u_{c1}$ and $f = f_{c1}$.

For $f = f_0 = \mathbf{const}$ this model coincide with the original kinetic model for the class of step type solutions $f^1(t,x,u) = \big(\theta(u-v_0) - \theta(u-v_1)\big) f_0$.
Setting $f = \mathbf{const}$ and neglecting the first term in the equation of state $p = p(n,f)$, we obtain the cold plasma model

$$n_t + (vn)_x = 0, \quad (vn)_t + (v^2 n + bn)_x = 0$$

which follows from original model in the case $f^1 = n(t,x)\delta(u - v(t,x))$.
It is easy to see that the kinetic model corresponds to the following infinite chain

$$\frac{\partial A_k}{\partial t} + \frac{\partial A_{k+1}}{\partial x} + bk\frac{A_{k-1}}{A_0}\frac{\partial A_0}{\partial x} = 0, \qquad (k = 0, 1, 2 \ldots)$$

The "water-bag" concept known in plasma physics allows one to obtain partial solutions described by closed system equations. We represent the k-th moment in the form

$$A_k = \frac{1}{k+1}\sum_{i=0}^{M} \varepsilon_i u_i^{k+1}, \qquad \sum_{i=0}^{M} \varepsilon_i = 0$$

Substitution of the moments into the chain yields a closed system of $M+1$

equations for the unknown functions $u_i(t,x)$:

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x}\Big(\frac{u_i^2}{2} + b\ln\sum_{k=0}^{M}\varepsilon_k u_k\Big) = 0 \tag{10}$$

System (10) coincides with the system of conservation laws (9) for the special class of solutions $f_{ci} = f_i = \mathrm{const}$. Note that by using the following representation of the solution (step function):

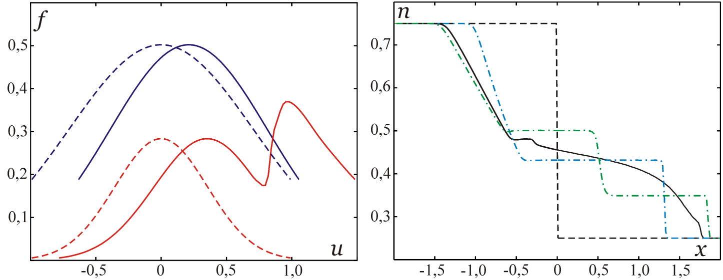
$$f^1 = \sum_{i=1}^{M} f_i\big(\theta(u-u_{i-1})-\theta(u-u_i)\big) = \sum_{i=0}^{M}\varepsilon_i\theta(u-u_i), \quad f_{i+1} = \sum_{k=0}^{i}\varepsilon_i$$

system (10) can be obtained directly from the kinetic model.

Numerical Simulation of Plasma Flow into a Plasma. Let us perform a test calculation of the decay of an initial discontinuity using the differential approximation (9). Suppose that $v_1(0,x) = -v_0(0,x) = 1$, and

$$f\Big|_{t=0} = \begin{cases} f_1(u) = a_1\exp(-a_3u^2), & x<0 \\ f_2(u) = a_2\exp(-a_4u^2), & x>0 \end{cases}$$

It has been shown by Gurevich, Pitaevskii that during evolution, the solution can undergo a kinetic turnover (the formation of two peaks of the distribution function which originally had a single peak).



Interection of plasma flows which have different temperatures ($a_4/a_3 = 4$) and densities ($n_1/n_2 = 3$). The plots (distribution function f and density n) are obtained for: $x \in [-2, 2]$; $M = 100$ (the system of 201 equations); $N = 200$ (number of cells in x); "flat" boundary conditions (perturbations do not reach the boundaries); $\mathbf{CFL} = 0.475$ and $b = 1$. The calculations are performed using Nessyahu–Tadmor central scheme. Dashed lines $t = 0$, solid lines $t = 1$; dash-dotted lines — results of cold plasma and gas-dynamic models.

Conclusion

We study the nonlinear mathematical models of shear flows of ideal liquid in open channels and the kinetic equations of quasineutral collisionless plasma. Emphasis is placed on the characteristic properties of the equations of motion, the construction of exact solutions and their physical interpretation.

Theoretical analysis of these models is based on the proposed by Teshukov concept of hyperbolicity and characteristics for systems of equations with operator coefficients.

A distinctive feature of integrodifferential models is the presence of both discrete and continuous spectrum of characteristic velocities. This is due to the fact that disturbances in shear flows are transmitted through the surface and internal waves.

THANK YOU FOR ATTENTION