From Braided Geometry to Integrable systems

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Plan

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Consider the Poisson-Lie bracket on $sl(2)^*$:

$$\{h, x\}_{PL} = 2x, \ \{h, y\}_{PL} = -2y, \ \{x, y\}_{PL} = h$$

and varieties $\mathcal{O}_\alpha$ defined by $Cas = 2^{-1}\, h^2 + xy + yx = \alpha$.
Also, consider the following quadratic bracket

$$\{h, x\}' = 2xh, \ \{h, y\}' = -2yh, \ \{x, y\}' = h^2.$$

These two brackets generate a Poisson pencil

$$\{\ ,\ \}_{a,b} = a\{\ ,\ \}_{PL} + b\{\ ,\ \}'.$$

Moreover, the element Cas is Poisson central for any bracket from this pencil:

$$\{Cas, f\}_{a,b} = 0, \ \forall f \in \mathbb{K}[sl(2)^*].$$

Thus, this Poisson pencil can be restricted to the quotient algebra

$$\mathbb{K}[sl(2)^*]/\langle Cas - \alpha\rangle, \ \alpha \in \mathbb{K}.$$

Let us quantize the P.p. $\{\,,\,\}_{a,b}$ and its restrictions to $\mathcal{O}_\alpha$.
As a quantum counterpart of the PL bracket we consider the enveloping algebra $U(sl(2)_\hbar)$ of $sl(2)_\hbar$ with multiplication table

$$[h,x]=2\hbar\, x,\ \ [h,y]=-2\hbar\, y,\ \ [x,y]=\hbar\, h.$$

Note that the element Cas remains central in $U(sl(2)_\hbar)$.
Now, quantize the bracket $\{\,,\,\}'$ alone in a somewhat elementary way. By replacing the Poisson bracket by the commutator and representing the r.h.s. in the symmetric form we get

$$hx-xh=\nu(hx+xh),\ hy-yh=-\nu(hx+xh),\ xy-yx=\nu h^2.$$

By putting $q^2=(1-\nu)/(1+\nu)$ we arrive to

$$q^2hx=xh,\ hy=q^2yh,\ (1+q^2)(xy-yx)=(1-q^2)h^2.$$

Denote this algebra $A(q)$. It has "good deformation property". This means that for a generic q

$$\dim A(q)^{(k)} = \dim \mathbb{K}[sl(2)^*]^{(k)}, \;\; k = 0, 1, 2, 3...$$

Observe that the algebra $A(q)$ is $U_q(sl(2))$-covariant.
In order to quantize the whole P.p. $\{\,,\,\}_{a,b}$ we consider the algebra $A(q, \hbar)$ defined by

$$q^2hx - xh = 2\hbar x,\; hy - q^2yh = -2\hbar y,$$

$$(1 + q^2)(xy - yx) - (1 - q^2)h^2 = 2\hbar h.$$

It is possible to see that $Gr\, A(q, \hbar) \cong A(q)$. So it is a two parameter deformation of the algebra $\mathbb{K}[sl(2)^*]$ indeed. Its semiclassical counterpart is the P.p. above.

In order to explicitly quantize this P.p. restricted to a variety $\mathcal{O}_\alpha$ we have to find the center of the algebra $A(q,\hbar)$.
Observe that the element Cas is not central in the algebra $A(q,\hbar)$ any more. However, the element

$$Cas_q = q^{-1}xy + qyx + (q+q^{-1})^{-1}h^2$$

is.
This element is not symmetric. Consequently, the pairing on the space $span(x,h,y)$ which is defined by the matrix inverse to that composed from the coefficients of the element Cas_q becomes

$$\langle h,h\rangle = 2_q = q+q^{-1},\ \langle x,y\rangle = q^{-1},\ \langle y,x\rangle = q.$$

It is not symmetric either. However, it is $U_q(sl(2))$-covariant.

Now, present the quantum counterpart of the restricted P.p. via the following quotient

$$A(q,\hbar)/\langle Cas_q - \alpha \rangle.$$

It is a braided non-commutative affine algebraic variety (hyperboloid).
In a similar way other "braided varieties" can be constructed. To this end we have to define "braided analogs" of the enveloping algebras $U(gl(n))$ or those $U(sl(n))$ and to find reasonable analogs of the equation $Cas_q = \alpha$.
Now, explain the meaning of the term "braided".

By braided geometry we mean geometry dealing with a braiding playing the role of a flip (or super-flip).
By a braiding we mean an invertible operator $R : V^{\otimes 2} \to V^{\otimes 2}$ where V is a vector space over the ground field $\mathbb{K}$ satisfying the so-called quantum Yang-Baxter equation

$$R_{12}\, R_{23}\, R_{12} = R_{23}\, R_{12}\, R_{23}, \quad R_{12} = R \otimes I, \;\; R_{23} = I \otimes R.$$

All notions and operators of "braided geometry" are associated to a given braiding.
First, we discuss a possible form of such a braiding.

The most studied are braidings of three types:

1. Involutive symmetries, i.e. such that $R^2 = I$.
2. Hecke symmetries, i.e. those subject to the Hecke condition

$$(q\,I - R)(q^{-1}\,I + R) = 0,\;\; q \in \mathbb{K}.$$

3. Birman-Murakami-Wenzl symmetries.

We are interested in Hecke symmetries.

The simplest examples are as follows. By fixing a basis $\{x, y\} \in V$ and the corresponding basis $\{x\otimes x, x\otimes y, y\otimes x, y\otimes y\}$ in $V^{\otimes 2}$ we represent Hecke symmetries R by matrices

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix}.$$

They are deformations of a usual flip and a super-flip respectively. We call them and their higher analogs (super-)standard. However, there is a lot of Hecke symmetries which are deformations neither of flips nor of super-flips.

To any Hecke symmetry R we associate "symmetric"

$$Sym(V) = T(V)/\langle Im(qI - R)\rangle$$

and "skew-symmetric"

$$\bigwedge(V) = T(V)/\langle Im(q^{-1}I + R)\rangle$$

algebras of the space V. They are graded algebras. Consider their Poincaré-Hilbert (PH) series:

$$\mathcal{P}_+(t) = \sum \dim\, Sym^k(V)t^k, \ \mathcal{P}_-(t) = \sum \dim \bigwedge^k(V)t^k.$$

For them the following relation was shown by myself 25 years ago

$$\mathcal{P}_+(t)\,\mathcal{P}_-(-t) = 1.$$

Note that in general, the PH series $\mathcal{P}_\pm(t)$ are always rational functions (Phung Ho Hai, A.Davydov). In a sense the couple $(p|r)$ where p is the degree of the numerator and r is that of the denominator of the function $\mathcal{P}_-(t)$, is an analog of the super-dimension. It is called *bi-rank* of R.

If $\mathcal{P}_-(t)$ is a polynomial, we say that the Hecke symmetry R is *even*. Then its bi-rank is $(p|0)$. In this case we call p *rank* of R.

For the classical flips and all their deformations $p = n = \dim V$ but in general it is not so.

Example: $\mathcal{P}_-(t) = 1 + n\,t + t^2,\ \ n > 2.$

Recently we have proved the *mounting property*.

For some Hecke symmetries $R : V^{\otimes 2} \to V^{\otimes 2}$, called skew-invertible an analog of the usual trace

$$Tr_R : \mathrm{End}(V) \to \mathbb{K}$$

can be introduced.
Thus, for the standard symmetry above such R-trace is

$$Tr_R \begin{pmatrix} a & b \\ c & d \end{pmatrix} = q^{-3}a + q^{-1}d.$$

Moreover, for a skew-invertible symmetry of bi-rank $(m|n)$ it is possible to contract a category of linear spaces looking like that of $U(gl(m|n))$-modules. This category contains the dual space V^*, all tensor product $V^{\otimes k} \otimes (V^*)^{\otimes l}$ and some their subspaces.
A space V^* is called dual if there exists a nondegenerated pairing $V \otimes V^* \to \mathbb{K}$ which is a category morphism. Whereas all category morphisms are assumed to commute with braidings.

We would like to define $Sym(\mathrm{End}(V))$ and $\bigwedge(\mathrm{End}(V))$ with g.d.p.
Note that only few objects of the mentioned category allow such "symmetric" and "skew-symmetric" algebras.
Fortunately, for the object $\mathrm{End}(V)$ its "symmetric" and "skew-symmetric" algebras with g.d.p. exist. By miracle $Sym(\mathrm{End}(V))$ is nothing but the so-called Reflection Equation (RE) algebra. It is a unital algebra generated by entries of a matrix $L = (l_i^j)$ subject to the system

$$R\, L_1\, R\, L_1 = L_1\, R\, L_1\, R, \;\; L_1 = L \otimes I.$$

It is a particular case of the so-called Quantum Matrix (QM) algebras. Another example is an "RTT algebra". It is a unital algebra generated by entries of a matrix $T = (t_i^j)$ subject to the system

$$R\, T_1\, T_2 = T_1\, T_2\, R, \;\; T_1 = T \otimes I, \; T_2 = I \otimes T.$$

Example: let R be standard Hecke symmetry above. By denoting $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we get the system

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix} =$$

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

Whereas the system for the corresponding RE algebra is

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \cdot$$

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \cdot$$

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

Compare properties of these two QM algebras. RTT algebra can be equipped with a bi-algebra structure. Essentially, this means that in it there is a coproduct Δ compatible with its product

$$\Delta(f \circ g) = \Delta(f) \circ \Delta(g).$$

If a Hecke symmetry is "even", in the RTT algebra there is an analog $det_R T$ of the determinant. This is a "group-like" element:

$$\Delta(det_R T) = det_R T \otimes det_R T.$$

If it is central, then the algebra $RTT/\langle det_R T - 1\rangle$ is an analog of the coordinate algebra $\mathbb{K}[SL(n)]$. Its semiclassical counterpart is the so-called Sklyanin bracket on $SL(n)$.

As for a RE algebra, being equipped with a similar coproduct it becomes "braided bi-algebra" (Majid). Essentially, this means

$$\Delta(f\circ g) = (\circ\otimes\circ)R_{23}(\Delta(f)\otimes\Delta(g)).$$

If R is even, in this algebra there is an analog $det^R L$ of the determinant as well. The quotient $REA/\langle det^R L - 1\rangle$ is another analog of the algebra $\mathbb{K}[SL(n)]$.
Note the the both algebras are graded quadratic with g.d.p.; they are two deformations of $\mathbb{K}[Mat(n)]$. However, their properties differ drastically.
In our "Braided Geometry" we only use RE algebras. We need RTT ones only for checking that our constructions are covariant w.r.t the coaction $\Delta(L) = T\otimes L\otimes T^{-1}$.

- The center of the RE algebra is much bigger than that of the RTT algebra. In particular, the elements $Tr_R L^k$ are central in the RE algebra, and those $Tr_R T^k$ are not in the RTT one.
- Consider a quadratic-linear algebra

$$R\,L_1\,R\,L_1 - L_1\,R\,L_1\,R = \hbar(R\,L_1 - L_1\,R).$$

This algebra is called modified RE algebra. It can be treated as a "braided" analog of the enveloping algebra $U(gl(m)_\hbar)$ (or $U(gl(m|n)_\hbar)$ and it turns into the latter algebra as $q \to 1$ provided R is standard (resp., super-standard).

Emphasize that for them an analog of the PBW theorem exists. By quotienting this algebra over $Tr_R L$ (which is central) we get a braided analog of the algebra $U(sl(m)_\hbar)$ or $U(sl(m|n)_\hbar)$.

The above algebra $A(q,\hbar)$ is nothing but a braided analog of the algebra $U(sl(2)_\hbar)$.

In general, standard modified RE algebras are quantum counterparts of similar Poisson pencils with $gl(m)$ type center.

Also note that for $q \neq 1$ an RE algebra and the corresponding modified RE algebra are isomorphic to each other. The isomorphism can be established by the shift map

$$L \to L - \frac{\hbar}{q-q^{-1}} I.$$

However, this isomorphism fails as $q = 1$.

One property more of the (modified or not) RE algebra is the following. For its generating matrix L there is an analog of the Cayley-Hamilton (CH) identity of the form

$$\sum_{i=0}^{p+r} a_{p+r-i}(L)\, L^i = 0$$

where $(p|r)$ is the bi-rank of R and the coefficients $a_i(L)$ are central in the algebra in question. The roots of the equation

$$\sum_{i=0}^{p+r} a_{p+r-i}(L)\, \mu^i = 0$$

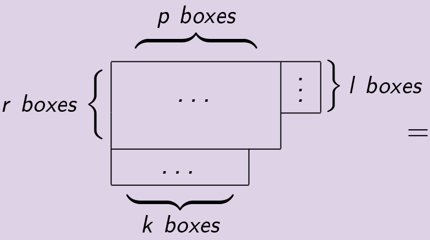
are analogs of the eigenvalues of a numerical matrix. But now they are treated to be elements of the algebraical extension of the center of this algebra.

Theorem

More precisely, this CH identity is

$$\sum_{i=0}^{p+r} L^{p+r-i} \sum_{k=\max\{0,i-r\}}^{\min\{i,p\}} (-1)^k\, q^{2k-i}\, s_{[p|r]^k_{i-k}}(L) = 0,$$

where s_λ is the Schur polynomial and



$$= \Big((p+1)^l, p^{(r-l)}, k\Big) =: [r|p]^l_k\,.$$

Theorem

This CH identity after being multiplied by $s_{[p|r]}$ factorizes as follows

$$\Big(\sum_{k=0}^{p}(-q)^k\, L^{p-k} s_{[p|r]^k}(L)\Big)\,\Big(\sum_{l=0}^{r} q^{-l}\, L^{r-l} s_{[p|r]_l}(L)\Big) = 0.$$

In terms of "even" roots μ_i and "odd" ones ν_i the identity becomes

$$(s_{[p|r]}(L))^2\,\prod_{i=1}^{p}(L-\mu_i I)\,\prod_{j=1}^{r}(L-\nu_j I) = 0.$$

Consider the simplest standard R and the corresponding modified RE algebra. Then the matrix $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is subject to

$$L^2 - (q\, Tr_R(L) + q^{-1}\hbar)L + (\frac{q^2}{2_q}(q\,(Tr_R\, L)^2 - Tr_R(L^2)) + \frac{q\hbar}{2_q} Tr_R(L))I = 0.$$

Remark.

For RTT algebras such a CH identity does not exist.
Observe that in the CH identities introduced by Gelfand and all coefficients are not scalar but diagonal matrices.

It is interesting to express different elements of the center of the (modified) RE algebra via eigenvalues μ_i. F.e. in the simplest standard case we have in the modified RE algebra the following

$$q^2\, Tr_R L^k = \mu_1^k\, \frac{q\mu_1 - q^{-1}\mu_2 - \hbar}{\mu_1 - \mu_2} + \mu_2^k\, \frac{q\mu_2 - q^{-1}\mu_1 - \hbar}{\mu_2 - \mu_1}.$$

Note that in the classical limit $\hbar = 0$, $q = 1$ this formula turns into

$$Tr\, L^k = \mu_1^k + \mu_2^k.$$

In general this formula is much more complicated (below, the eigenvalues μ_i and ν_i are respectively "even" and "odd"):

Theorem

In the REA the following holds

$$Tr_R L^k = \sum_{i=1}^{p} d_i \mu_i^k + \sum_{j=1}^{r} \widetilde{d}_j \nu_j^k, \text{ where}$$

$$d_i = q^{-1} \prod_{m=1,\, m\neq i}^{p} \frac{\mu_i - q^{-2}\mu_m}{\mu_i - \mu_m} \prod_{j=i}^{r} \frac{\mu_i - q^2 \nu_j}{\mu_i - \nu_j},$$

$$\widetilde{d}_j = -q \prod_{i=1}^{p} \frac{\nu_j - q^{-2}\mu_i}{\nu_j - \mu_i} \prod_{m=1,\, m\neq j}^{r} \frac{\nu_j - q^2 \nu_m}{\nu_j - \nu_m}.$$

Our next aim is to define elements of differential calculus on an RE algebra and to exhibit some applications. As a result we'll get a very astonishing differential calculus on the algebra $U(gl(m|n))$.

Let us modify the Woronowicz's differential calculus on a matrix pseudogroup. In fact, his calculus consists of an RTT algebra playing the role of "function algebra", an algebra generated by analogs of one-sided "vector fields" and that generated by first differentials. Woronowicz keeps the usual Leibniz rule for the de Rham operators and defines some permutation relations between "functions" and "differentials". Lately, the algebra generated by "vector fields" was identified as an RE one.

We replaced the RTT algebra by another copy of the RE algebra. Thus, we have two copies of the RE algebra. One of them (generated by the matrix M and denoted $\mathcal{M}$) is that of "functions", and the other one (generated by the matrix L and denoted $\mathcal{L}$) is that of "differential operators" (its generators are "vector fields"). The action of the algebra $\mathcal{L}$ onto that $\mathcal{M}$ is defined via permutation relations between the generating matrices

$$R\, L_1\, R\, M_1 = M_1\, R\, L_1\, R^{-1}.$$

Such permutation relations enable us to define an actions of "vector fields" l_i^j onto "functions". In order to get the action $l_i^j(m_k^l....m_p^q)$ we have to transpose the "vector field" to the extreme right position via the permutation relations and to apply to it the counit

$$\varepsilon(1) = 1,\ \varepsilon(l_i^j) = \delta_i^j.$$

Thus, elements l_i^j are treated to be analogs of left (right-invariant) vector fields. However, they are "multiplicative" vector fields, i.e. they are based on the group-like coproduct

$$\Delta(l_i^j) = \sum_p l_i^p \otimes l_p^j.$$

In order to get vector fields which are more similar to the classical ones, we pass to the modified form of the RE algebra $\mathcal{L}$ (i.e. we apply the shift $L = K - \frac{1}{q-q^{-1}} I$). Then the coproduct on the generators k_i^j of the modified form of the RE algebra is

$$\Delta(k_i^j) = k_i^j \otimes 1 + 1 \otimes k_i^j - (q - q^{-1}) \sum_p k_i^p \otimes k_p^j.$$

It becomes classical in the limit $q \to 1$.

Now, by considering the matrix $D = M^{-1}\,K$ we get analogs of partial derivatives on the algebra $\mathcal{M}$. We call the finial algebra *braided Weyl algebra* and denote it $W(L, D)$. Its defining relations are

$$R\,M_1\,R\,M_1 - M_1\,R\,M_1\,R = 0,$$

$$R^{-1}\,D_1\,R^{-1}\,D_1 - D_1\,R^{-1}\,D_1\,R^{-1} = 0,$$

$$D_1\,R\,M_1\,R - R\,M_1 R^{-1}\,D_1 = R.$$

The entries ∂_i^j of the matrix D are called *braided partial derivatives*.

Now, by passing from the matrix M to the generating matrix N of the corresponding modified RE algebra we get (after a slight renormalization) the following system

$$\begin{aligned} R\,N_1 R\,N_1 - N_1 R\,N_1 R &= \hbar\,(R\,N_1 - N_1 R) \\ R^{-1} D_1 R^{-1} D_1 &= D_1 R^{-1} D_1 R^{-1} \\ D_1 R\,N_1 R - R\,N_1 R^{-1} D_1 &= R + \hbar\, D_1 R\,. \end{aligned}$$

It also defines a *braided Weyl algebra* denoted $W(N, D)$. But this algebra is well defined on a modified RE algebra.

By assuming R to be super-standard and by passing to the limit $q \to 1$ we get differential calculus on $U(gl(p|r)_\hbar)$.

Let us consider an example: $p = 2, r = 0$. Denote a, b, c, d the standard generators of the algebra $U(gl(2)_\hbar)$ such that

$$[a, b] = \hbar\, b,\ [a, c] = -\hbar\, c,\ [a, d] = 0, \ldots\ldots,\ [d, c] = \hbar\, c.$$

Also, pass to generators of the compact form, namely, $U(u(2)_\hbar)$

$$t = \frac{1}{2}(a + d),\ x = \frac{i}{2}(b + c),\ y = \frac{1}{2}(c - b),\ z = \frac{i}{2}(a - d)$$

we get the standard $u(2)_\hbar$ table of commutators

$$[x,\, y] = \hbar\, z,\ [y,\, z] = \hbar\, x,\ [z,\, x] = \hbar\, y,\ t \text{ is central.}$$

Then the corresponding permutation relations become

$$[\partial_t, t] = \frac{\hbar}{2}\partial_t + 1,\ [\partial_t, x] = -\frac{\hbar}{2}\partial_x,\ [\partial_t, y] = -\frac{\hbar}{2}\partial_y,\ [\partial_t, z] = -\frac{\hbar}{2}\partial_z,$$

$$[\partial_x, t] = \frac{\hbar}{2}\partial_x,\ [\partial_x, x] = \frac{\hbar}{2}\partial_t + 1,\ [\partial_x, y] = \frac{\hbar}{2}\partial_z,\ [\partial_x, z] = -\frac{\hbar}{2}\partial_y,$$

$$[\partial_y, t] = \frac{\hbar}{2}\partial_y,\ [\partial_y, x] = -\frac{\hbar}{2}\partial_z,\ [\partial_y, y] = \frac{\hbar}{2}\partial_t + 1,\ [\partial_y, z] = \frac{\hbar}{2}\partial_x,$$

$$[\partial_z, t] = \frac{\hbar}{2}\partial_z,\ [\partial_z, x] = \frac{\hbar}{2}\partial_y,\ [\partial_z, y] = -\frac{\hbar}{2}\partial_x,\ [\partial_z, z] = \frac{\hbar}{2}\partial_t + 1.$$

The Leibnitz rule on this algebra can be presented via the coproduct

$$\Delta\partial_i^j = \partial_i^j \otimes 1 + 1 \otimes \partial_i^j + \hbar \sum_p \partial_p^j \otimes \partial_i^p.$$

Whereas the partial derivatives commute with each other.

Thus, on the algebra $U(u(2)_\hbar)$ we can define an analog of any differential operator and equation. F.e. the Klein-Gordon equation is defined in the classical way

$$(\square - m^2)\, f = 0, \;\; \square = \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2$$

Here f is an element of the algebra $U(u(2)_\hbar)$ or its completion and m is the "mass of a NC particle". However, the partial derivatives coming in this equation are subject to the modified version of the Leibnitz rule. In a similar way we define NC analogs of other wave (Dirac, Maxwell...) operators.

As for the de Rham operator d it can be defined on "functions" via

$$d(f) = dt\,\partial_t(f) + dx\,\partial_x(f) + dy\,\partial_y(f) + dz\,\partial_z(f).$$

In a similar way we can define de Rham operator d on differential forms. Namely, we put

$$d(\omega\, f) = \omega\, dt\,\partial_t(f) + \omega\, dx\,\partial_x(f) + \omega\, dy\,\partial_y(f) + \omega\, dz\,\partial_z(f)$$

where ω is a pure differential form ($d\,t, ...d\,t\,dx, ...$ and so on). The relations between the differentials $dt, ..., dz$ are assumed to be classical $dt\,dx = -dx\,dt$, i.e. these generators anticommute. This property together with the commutativity of the partial derivatives entails $d^2 = 0$, i.e. d is a differential indeed.

Our immediate objective is to define and calculate the radial part of the Laplacian $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ on the algebra $U(u(2)_\hbar)$.

Definition

The operator Δ restricted to the center $Z = Z(U(u(2)_\hbar))$ of the algebra $U(u(2)_\hbar)$ is called the radial part of the Laplacian Δ and is denoted Δ_{rad}.

This definition is motivated by the following theorem

Theorem

The result of applying Δ to any element $Tr\, L^k$ is central as well.

Note that $t,\ x^2 + y^2 + z^2 \in Z(U(u(2)_\hbar))$. By expressing these elements and those $\Delta(t)$, $\Delta(x^2 + y^2 + z^2)$ via μ_1, μ_2, we get Δ_{rad} realized through these eigenvalues.

By introducing new variables $\lambda = \mu_1 + \mu_2$ and $\mu = (\mu_1 - \mu_2)^2$ we get

$$\Delta_{rad}\,(f(\lambda,\mu)) = \frac{1}{\hbar^2}\,(2f(\lambda+2\hbar,\mu)$$

$$-f(\lambda+2\hbar,\mu+4\hbar^2+4\hbar\sqrt{\mu}) - f(\lambda+2\hbar,\mu+4\hbar^2-4\hbar\sqrt{\mu})$$

$$+\frac{2}{\sqrt{\mu}}\,\frac{1}{\hbar}\,(f(\lambda+2\hbar,\mu+4\hbar^2-4\hbar\sqrt{\mu})$$

$$-f(\lambda+2\hbar,\mu+4\hbar^2+4\hbar\sqrt{\mu})).$$

In the limit $\hbar \to 0$ the difference operator Δ_{rad} turns into the following second order differential operator

$$-16\mu\frac{d^2}{d\,\mu^2} - 24\frac{d}{d\,\mu}.$$

Being rewritten via the variable r such that $\mu = -4r^2$ we get the usual radial part of the classical Laplacian on $\mathbb{R}^3$

$$\frac{d^2}{d\,r^2} + \frac{2}{r}\frac{d}{d\,r}.$$

In the classical (commutative) setting the radial part of the Laplacian

$$\Delta = TrD^2 = \sum_{1\le i,j\le m} \partial_{l_i^j}\partial_{l_j^i}$$

defined on $\mathbb{K}[Mat(m)]$ equals

$$\sum \partial_i^2 + 2\sum_{i,j} \frac{\partial_i - \partial_j}{\mu_i - \mu_j}, \quad \partial_i = \partial_{\mu_i}.$$

This operator is gauge equivalent to the Calogero-Moser one

$$\sum \partial_i^2 + 2\sum_{i<j} \frac{1}{(\mu_i - \mu_j)^2}.$$

Discuss a way of getting a two parameter deformation of this model. Let $\mathcal{N}$ be the standard mRE algebra (it is a braided deformation of the enveloping algebra $U(gl(m)_\hbar))$. Also, let D be the matrix of the partial derivatives on this mRE algebra. Consider the operators $Tr_R D^k, k = 0, 1, 2, ..., m$ acting on the algebra $\mathcal{N}$.
They commute with each other. Besides, they map the center $Z = Z(U(gl(m)_\hbar))$ of the algebra $U(gl(m)_\hbar)$ into itself.

Consequently, restrictions of the operators $Tr_R D^k$ to Z are well defined.

By expressing these restricted operators via the eigenvalues of the generating matrix N of the algebra $\mathcal{N}$ we get a family of operators in involution. Hopefully, these operators are difference ones and they are two-parameter deformations of the corresponding classical differential operators which are gauge equivalent to the rational Calogero-Moser operator and its higher counterparts respectively.

However, computations in higher dimensional case become much harder.