Relations between Toda and KdV-type hierarchies

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Outline of the talk

- Compatible Lax equations for f.d. matrices
- $\mathbb{Z} \times \mathbb{Z}$-matrices
- Related hierarchies
- Pseudo differential operators
- Associated hierarchies
- Solutions+relations

Compatible Lax equations for f.d. matrices 1

- Matrix Lie group G with Lie algebra $\mathfrak{g}$
- $\mathfrak{g}_i, i = 1, 2,$ Lie subalgebras of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

- π_i the projection of $\mathfrak{g}$ onto $\mathfrak{g}_i$ induced by this decomposition
- $\mathfrak{g}_i$ Lie algebras of the Lie subgroup G_i

$$G = G_1 G_2, \text{ with } G_1 \cap G_2 = \mathrm{Id}$$

- $g \in G$, $g = g_1 g_2, g_i \in G_i$
- Two sets linear independent, commuting matrices:

$$\{F_i \mid 1 \leq i \leq m_2\} \in \mathfrak{g}_2 \text{ and } \{G_j \mid 1 \leq j \leq m_1\} \in \mathfrak{g}_1.$$

- $[F_i, G_j] = 0$ for all i and j.

Compatible Lax equations for f.d. matrices 2

- They generate the commuting flows

$$\gamma = \gamma(t,s) := \gamma(t_i, s_j) = \exp(\sum_{i=1}^{m_2} t_i F_i + \sum_{j=1}^{m_1} s_j G_j)$$

- $g \in G$:

$$\gamma(t,s) g \gamma(t,s)^{-1} = g_1(t,s)^{-1} g_2(t,s).$$

- Multidimensional flows $\mathcal{F}_i$ and $\mathcal{G}_j$ in $\mathfrak{g}$:

$$\mathcal{F}_i := g_1 F_i g_1^{-1}, 1 \leq i \leq m_2, \text{ and } \mathcal{G}_j := g_2 G_j g_2^{-1}, 1 \leq i \leq m_1.$$

- This deformation preserves the commutativity of each set

$$[\mathcal{F}_{i_1}, \mathcal{F}_{i_2}] = 0 = [\mathcal{G}_{j_1}, \mathcal{G}_{j_2}],$$

Compatible Lax equations for f.d. matrices 3

- $G_1 + G_2$-variant:

Theorem

Notations being as above, the deformations $\{\mathcal{F}_i\}$ *and the* $\{\mathcal{G}_j\}$ *of the initial commuting directions satisfy*

$$\frac{\partial}{\partial t_{i_1}}(\mathcal{F}_{i_2}) = [\mathcal{F}_{i_2}, \pi_1(\mathcal{F}_{i_1})] = [\pi_2(\mathcal{F}_{i_1}), \mathcal{F}_{i_2}]$$

$$\frac{\partial}{\partial s_{j_1}}(\mathcal{G}_{j_2}) = [\mathcal{G}_{j_2}, \pi_2(\mathcal{G}_{j_1})] = [\pi_1(\mathcal{G}_{j_1}), \mathcal{G}_{j_2}]$$

$$\frac{\partial}{\partial s_{j_1}}(\mathcal{F}_{i_2}) = [\pi_1(\mathcal{G}_{j_1}), \mathcal{F}_{i_2}]$$

$$\frac{\partial}{\partial t_{i_1}}(\mathcal{G}_{j_2}) = [\pi_2(\mathcal{F}_{i_1}), \mathcal{G}_{j_2}].$$

Compatible Lax equations for f.d. matrices 4

- G_1-variant: only the $\{F_i \mid 1 \leq i \leq m_2\} \in \mathfrak{g}_2$
- Commuting flows:

$$\gamma(t) = \gamma(t_1, \cdots .t_m) = \exp(\sum_{i=1}^{m} t_i F_i)$$

- Decomposition:

$$\gamma(t) g \gamma(t)^{-1} = g_1(t)^{-1} g_2(t).$$

Theorem

The deformations $\{\mathfrak{F}_i := g_1 F_i g_1^{-1}\}$ of the initial commuting directions satisfy

$$\frac{\partial}{\partial t_{i_1}}(\mathfrak{F}_{i_2}) = [\mathfrak{F}_{i_2}, \pi_1(\mathfrak{F}_{i_1})] = [\pi_2(\mathfrak{F}_{i_1}), \mathfrak{F}_{i_2}]$$

$\mathbb{Z} \times \mathbb{Z}$-matrices 1

- Commutative k-algebra R, $k = \mathbb{R}$ or $\mathbb{C}$.
- $M_{\mathbb{Z}}(R)$: $\mathbb{Z} \times \mathbb{Z}$-matrices, coefficients from R
- $A = (a_{ij}) \in M_{\mathbb{Z}}(R)$:

$$A = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathbf{a_{n-1\ n-1}} & a_{n-1\ n} & a_{n-1\ n+1} & \ddots \\ \ddots & a_{n\ n-1} & \mathbf{a_{n\ n}} & a_{n\ n+1} & \ddots \\ \ddots & a_{n+1\ n-1} & a_{n+1\ n} & \mathbf{a_{n+1\ n+1}} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

$\mathbb{Z} \times \mathbb{Z}$-matrices 2

- To $\{d(s) | s \in \mathbb{Z}\}$ in R is associated diag$(d(s))$:

$$\begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathbf{d(n-1)} & 0 & 0 & \ddots \\ \ddots & 0 & \mathbf{d(n)} & 0 & \ddots \\ \ddots & 0 & 0 & \mathbf{d(n+1)} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

- Diagonal matrices:

$$\mathcal{D}_1(R) = \{d = \operatorname{diag}(d(s)) | d(s) \in R \text{ for all } s \in \mathbb{Z}\}.$$

$\mathbb{Z} \times \mathbb{Z}$-matrices 3

- Shift matrix Λ^{-1}

$$\Lambda^{-1} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathbf{0} & 0 & 0 & \ddots \\ \ddots & 1 & \mathbf{0} & 0 & \ddots \\ \ddots & 0 & 1 & \mathbf{0} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

- Action of the $\{\Lambda^m \mid m \in \mathbb{Z}\}$ on $\mathcal{D}_1(R)$:

$$\Lambda^m \mathrm{diag}(d(s))\Lambda^{-m} = \mathrm{diag}(d(s+m)).$$

$\mathbb{Z} \times \mathbb{Z}$-matrices 4

- Each $A = (a_{ij}) \in M_{\mathbb{Z}}(R)$: decomposes uniquely

$$A = \sum_{i \in \mathbb{Z}} d_i \Lambda^i, d_i \in \mathcal{D}_1(R)$$

- Lower triangular matrices

$$LT(R) = \{L \mid L = \sum_{i \leq N} \ell_i \Lambda^i, \ell_i \in \mathcal{D}_1(R), N \in \mathbb{Z}\}$$

- Upper triangular matrices

$$UT(R) = \{U \mid U = \sum_{i \geq N} u_i \Lambda^i, u_i \in \mathcal{D}_1(R), N \in \mathbb{Z}\}$$

- Difference operator $\Delta := \Lambda^{-1} - \mathrm{Id}$:

$$UT(R) = \{U \mid U = \sum_{i \leq N} d_i \Delta^i, d_i \in \mathcal{D}_1(R), N \in \mathbb{Z}\}$$

$\mathbb{Z} \times \mathbb{Z}$-matrices 5

- Consider

$$w_1 = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathbf{0} & 0 & 1 & \ddots \\ \ddots & 0 & \mathbf{1} & 0 & \ddots \\ \ddots & 1 & 0 & \mathbf{0} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

- Action on diagonal and shift matrices:

$$w_1 \mathrm{diag}(d(j)) w_1 = \mathrm{diag}(d(-j)) \text{ and } w_1 \Lambda^m w_1 = \Lambda^{-m}.$$

- $LT(R)$ and $UT(R)$ isomorphic: $L \mapsto w_1 L w_1$

Decompositions 1

- Two relevant decompositions in $LT(R)$
- First: $L = \sum_{i \leq N} \ell_i \Lambda^i \in LT(R)$, $L = L_{<0} + L_{\geq 0}$

$$L_{<0} = \sum_{i<0} \ell_i \Lambda^i, L_{\geq 0} = \sum_{i \geq 0} \ell_i \Lambda^i$$

- $LT(R) = LT(R)_{<0} \oplus LT(R)_{\geq 0}$. Hence

$$\mathfrak{g}_1 = LT(R)_{<0}, \ \ \mathfrak{g}_2 = LT(R)_{\geq 0}$$

- $G_1 = U_-$ group associated with $\mathfrak{g}_1$

$$U_- = \{g = \mathrm{Id} + \sum_{i<0} g_i \Lambda^i, g_i \in \mathcal{D}_1(R)\}$$

- $G_2 = ?$

Decompositions 2

- Second: $L = \sum_{i \leq N} \ell_i \Lambda^i \in LT(R)$, $L = L_{\leq 0} + L_{>0}$

$$L_{\leq 0} = \sum_{i \leq 0} \ell_i \Lambda^i, L_{>0} = \sum_{i>0} \ell_i \Lambda^i$$

- $LT(R) = LT(R)_{\leq 0} \oplus LT(R)_{>0}$. Hence

$$\mathfrak{g}_1 = LT(R)_{\leq 0}, \ \ \mathfrak{g}_2 = LT(R)_{>0}$$

- $G_1 = P_-$ group associated with $\mathfrak{g}_1$

$$P_- = \{g = \sum_{i \leq 0} g_i \Lambda^i, g_i \in \mathcal{D}_1(R), g_0 \in \mathcal{D}_1(R)^*\}$$

- $G_2 = ?$

Decomposition 3

- Equivalent of last decomposition in $UT(R)$
- $M = \sum_{i \geq N} m_i \Lambda^i \in UT(R)$, $M = M_{\geq 0} + M_{<0}$

$$M_{\geq 0} = \sum_{i \leq 0} m_i \Lambda^i, M_{<0} = \sum_{i > 0} \ell_i \Lambda^i$$

- $UT(R) = UT(R)_{\geq 0} \oplus UT(R)_{<0}$. Hence

$$\mathfrak{g}_1 = UT(R)_{\geq 0}, \ \ \mathfrak{g}_2 = UT(R)_{><0}$$

- $G_1 = P_+$ group associated with $\mathfrak{g}_1$

$$P_+ = \{g = \sum_{i \geq 0} g_i \Lambda^i, g_i \in \mathcal{D}_1(R), g_0 \in \mathcal{D}_1(R)^*\}$$

- $G_2 = ?$

Decomposition 3

- Equivalent of last decomposition in $UT(R)$:
- $M = \sum_{i \geq N} m_i \Lambda^i \in UT(R)$, $M = M_{\geq 0} + M_{<0}$

$$M_{\geq 0} = \sum_{i \leq 0} m_i \Lambda^i, M_{<0} = \sum_{i > 0} \ell_i \Lambda^i$$

- $UT(R) = UT(R)_{\geq 0} \oplus UT(R)_{<0}$. Hence

$$\mathfrak{g}_1 = UT(R)_{\geq 0}, \ \ \mathfrak{g}_2 = UT(R)_{><0}$$

- $G_1 = P_+$ group associated with $\mathfrak{g}_1$

$$P_+ = \{g = \sum_{i \geq 0} g_i \Lambda^i, g_i \in \mathcal{D}_1(R), g_0 \in \mathcal{D}_1(R)^*\}$$

- $G_2 = ?$

Decomposition 4

- $M_{\mathbb{Z}}(R) = LT(R)_{<0} \oplus UT(R)_{\geq 0} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$
- $G_1 = U_-$ group associated with $\mathfrak{g}_1$
- $G_2 = P_+$ group associated with $\mathfrak{g}_2$

$LT(R)_{\geq 0}$-hierarchy

- Deformation of Λ in lower triangular matrices:

$$\mathcal{L} := \Lambda + \sum_{i \leq 0} l_i \Lambda^i$$

- Example: $\mathcal{L} = U \Lambda U^{-1}$, $U \in G_1 = U_-$.
- R ring of functions in flow parameters $\{t_i\}$ w.r.t. $\Lambda^i, i \geq 1$, stable under all

$$\partial_{t_i} := \frac{\partial}{\partial t_i}.$$

- Lax equations of the $LT(R)_{\geq 0}$-hierarchy:

$$\partial_{t_i}(\mathcal{L}) = [(\mathcal{L}^i)_{\geq 0}, \mathcal{L}]$$

- Trivial solution Λ

$LT(R)_{>0}$-hierarchy

- Deformation of Λ in lower triangular matrices:

$$\mathcal{N} := \sum_{i \leq 1} n_i \Lambda^i, n_1 \in \mathcal{D}_1(R)^*$$

- Example: $\mathcal{N} = P\Lambda P^{-1}$, $P \in G_1 = P_-$.
- R ring of functions in flow parameters $\{t_i\}$ w.r.t. $\Lambda^i, i \geq 1$, stable under all

$$\partial_{t_i} := \frac{\partial}{\partial t_i}.$$

- Lax equations of the $LT(R)_{>0}$-hierarchy:

$$\partial_{t_i}(\mathcal{N}) = [(\mathcal{N}^i)_{>0}, \mathcal{N}]$$

- Trivial solution Λ

$UT(R)_{<0}$-hierarchy

- Deformation of Λ^{-1} in upper triangular matrices:

$$\mathcal{M} := \sum_{i \geq -1} m_i \Lambda^i, m_{-1} \in \mathcal{D}_1(R)^*$$

- Example: $\mathcal{M} = P\Lambda^{-1}P^{-1}$, $P \in G_1 = P_+$.
- R ring of functions in flow parameters $\{s_j\}$ w.r.t. $\Lambda^{-j}, j \geq 1$, stable under all

$$\partial_{s_j} := \frac{\partial}{\partial s_j}.$$

- Lax equations of the $UT(R)_{<0}$-hierarchy:

$$\partial_{s_j}(\mathcal{M}) = [(\mathcal{M}^i)_{<0}, \mathcal{M}]$$

- Trivial solution Λ^{-1}

Two dimensional Toda hierarchy

- Two deformations

$$\mathcal{L} := \Lambda + \sum_{i \leq 0} l_i \Lambda^i, \text{ and } \mathcal{M} := \sum_{i \geq -1} m_i \Lambda^i, m_{-1} \in \mathcal{D}_1(R)^*$$

- R ring of functions in the flow parameters $\{t_i\}$ w.r.t. $\Lambda^i, i \geq 1$, and the flow parameters $\{s_j\}$ w.r.t. $\Lambda^{-j}, j \geq 1$, stable under all

$$\partial_{t_i} := \frac{\partial}{\partial t_i} \text{ and } \partial_{s_j} := \frac{\partial}{\partial s_j}.$$

- Lax equations of the two dimensional Toda hierarchy:

$$\partial_{t_i}(\mathcal{L}) = [(\mathcal{L}^i)_{\geq 0}, \mathcal{L}], \partial_{t_i}(\mathcal{M}) = [(\mathcal{L}^i)_{\geq 0}, \mathcal{M}]$$
$$\partial_{s_j}(\mathcal{M}) = [(\mathcal{M}^j)_{<0}, \mathcal{M}], \partial_{s_j}(\mathcal{L}) = [(\mathcal{M}^j)_{<0}, \mathcal{L}]$$

- Trivial solutions Λ, Λ^{-1}

Pseudo differential operators 1

- R ring of functions in $\{t_i \mid i \geq 1\}$
- $\partial_i = \frac{\partial}{\partial t_i} : R \rightarrow R$, priveleged derivation $\xi = \partial_1$
- $R[\xi] = \{\sum_{i=0}^{n} a_i \xi^i, a_i \in R \text{ for all } i \geq 0\}$
- Multiplication in $R[\xi]$:

$$\left(\sum_i a_i \xi^i\right)\left(\sum_j b_j \xi^i\right) = \sum_{i,j} \sum_{0 \leq k \leq i} \binom{i}{k} a_i \partial_1^k (b_j) \xi^{i+j-k},$$

Pseudo differential operators 2

- For each $m \in \mathbb{Z}$, $k \geq 1$,

$$\binom{m}{k} := \frac{m(m-1)\cdots(m-k+1)}{k!}, \binom{m}{0} := 1$$

- Pseudo differential operators

$$\mathrm{Psd} = R[\xi, \xi^{-1}) = \{p = \sum_{j=-\infty}^{N} p_j \xi^j, p_j \in R\},$$

- Multiplication:

$$a.b := \sum_j \sum_i \sum_{s=0}^{\infty} \binom{i}{s} a_i \partial_1^s(b_j) \xi^{i+j-s}$$

Decompositions in Psd 1

- First decomposition in $R[\xi, \xi^{-1})$:

$$P = \sum_j P_j \xi^j = \sum_{j<0} P_j \xi^j + \sum_{j\geq 0} P_j \xi^j = P_{<0} + P_{\geq 0}$$

- Lie algebra $\mathrm{Psd} = \mathrm{Psd}_{<0} \oplus \mathrm{Psd}_{\geq 0} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$
- Group corresponding to $\mathfrak{g}_1$

$$G_1 = \{g = 1 + \sum_{j<0} g_j \xi^j, g_j \in R\}$$

Decompositions in Psd 2

- Second decomposition in $R[\xi, \xi^{-1})$:

$$P = \sum_j P_j \xi^j = \sum_{j \leq 0} P_j \xi^j + \sum_{j > 0} P_j \xi^j = P_{\leq 0} + P_{>0}$$

- Lie algebra decomposition $\mathrm{Psd} = \mathrm{Psd}_{\leq 0} \oplus \mathrm{Psd}_{>0}$
- Group corresponding to $\mathfrak{g}_1$

$$G_1 = \{g = \sum_{j \leq 0} g_j \xi^j, g_j \in R, g_0 \in R^*\}$$

KP hierarchy

- Decomposition $\mathrm{Psd} = \mathrm{Psd}_{<0} \oplus \mathrm{Psd}_{\geq 0} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$
- Deformation $L = \xi + \sum_{i \geq 1} \ell_{i+1}\xi^{-i}$, $B_k = (L^k)_{\geq 0}$
- Examples: $L = P\xi P^{-1}$, $P \in G_1, P = \mathrm{Id} + \sum_{i\geq 1} p_i \xi^{-i}$
- R ring of functions in the parameters $\{t_i\}$ corresponding to $\xi^i, i \geq 1$, , stable under all

$$\partial_{t_i} := \frac{\partial}{\partial t_i}.$$

- Lax equations of the KP hierarchy

$$\partial_{t_{k_1}}(L^{k_2}) = [B_{k_1}, L^{k_2}] = [L^{k_2}, L^{k_1}_{<0}], k_1 \text{ and } k_2 \geq 1.$$

Strict KP hierarchy

- Decomposition $\mathrm{Psd} = \mathrm{Psd}_{\leq 0} \oplus \mathrm{Psd}_{>0} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$
- Consider deformations

$$M = \xi + m_1 + m_2\xi^{-1} + \cdots$$

- Examples: $M = P\xi P^{-1}$,
  $P \in G_1, P = p_0 + \sum_{i\geq 1} p_i\xi^{-i}, p_0 \in R^*$
- R and ∂_{t_i} as above
- Let $C_r = (M^r)_{>0}, r \geq 1$.
- Strict KP hierarchy for M and its powers:

$$\partial_{t_{k_1}}(M^{k_2}) = [C_{k_1}, M^{k_2}] = [M^{k_2}, M^{k_1}_{\leq 0}], k_1 \text{ and } k_2 \geq 1$$

Geometry 1

- Hilbert space

$$H = \{\sum_{n\in\mathbb{Z}} a_n z^n \mid a_n \in \mathbb{C}, \sum_{n\in\mathbb{Z}} \mid a_n \mid^2 < \infty\},$$

- Decomposition $H = H_+ \oplus H_-$, where

$$H_+ = \{\sum_{n\geq 0} a_n z^n \in H\} \quad \text{and} \quad H_- = \{\sum_{n<0} a_n z^n \in H\}$$

- Grassmannian $Gr(H)$: closed subspaces W of H such that
  - Orthogonal projection $p_+ : W \to H_+$ is Fredholm
  - Orthogonal projection $p_- : W \to H_-$ is Hilbert-Schmidt.

Geometry 2

- Connected components of $Gr(H)$: $\ell \in \mathbb{Z}$

$$Gr^{(\ell)}(H) = \left\{ W \in Gr(H) |\ p_+ : z^{-\ell} W \to H_+ \text{ has index zero} \right\}.$$

- $Gl_{res}^{(0)}(H)$ group of all bounded invertible operators $g : H \to H$ that decompose with respect to $H = H_+ \oplus H_-$ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with a and d Fredholm of index zero and b and c Hilbert–Schmidt.

- Each $Gr^{(\ell)}(H)$ is a homogeneous space for the group $Gl_{res}^{(0)}(H)$.

Geometry 3

- Commuting flows for KP, strict KP + lower triangular:

$$\Gamma_+ = \{\gamma_+(t) := \exp(\sum_{i\geq 1} t_i z^i) \mid \sum_{i\geq 1} |t_i| N^i < \infty, \text{ some } N > 1\}.$$

- For $UT(R)_{<0}$-hierarchy:

$$\Gamma_- = \{\gamma_-(s) := \exp(\sum_{j\geq 1} s_j z^{-j}) \mid \sum_{j\geq 1} |s_j| M^j < \infty, \text{ some } M > 1\}.$$

- For two dimensional Toda:

$$\Gamma = \{\gamma(t,s) = \gamma_+(t)\gamma_-(s) \mid \gamma_+ \in \Gamma_+, \gamma_- \in \Gamma_-\}$$

Geometry 4

- $\mathfrak{P}_\ell$ embeddings $w : z^\ell H_+ \to H$ such that w.r.t.

$$H = (z^\ell H_+) \oplus (z^\ell H_+)^\perp$$

$w = \binom{w_+}{w_-}$, with w_- Hilbert-Schmidt , $w_+ -$ Id trace class.

- $\mathfrak{P}_\ell$ fiber bundle over $Gr^{(\ell)}(H)$ with fiber

$$\mathfrak{T}_\ell = \{t \in \mathrm{Aut}(z^\ell H_+) | t - \mathrm{Id} \text{ is of trace class}\}.$$

- Extension Gl of $Gl_{res}^{(0)}(H)$ to lift action from $Gr^{(\ell)}(H)$ to $\mathfrak{P}_\ell$

$$Gl = \{(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, q) \in GL_{res}^{(0)}(H) \times \mathrm{Aut}(z^\ell H_+), aq^{-1} - \mathrm{Id} \text{ trace class}\}.$$

Geometry 5:

- Group Gl acts by $w \mapsto gwq^{-1}$ on $\mathfrak{P}_\ell$.
- Γ_+ embeds in a natural way into Gl

$$\gamma_+ = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto (\gamma_+, a).$$

- For $w \in \mathfrak{P}_\ell$, define $\tau_w : Gl \to \mathbb{C}$ by

$$\tau_w((g,q)) = \det((g^{-1}wq)_+).$$

- If t belongs to $\mathfrak{T}_\ell$, then there holds $\tau_{w \circ t} = \det(t)\tau_w$.
- The restriction of τ_w to Γ_+ is denoted $\tau_W((t_i)) = \tau_W(t)$.

Solutions KP

- Segal-Wilson:
- $W \in Gr^{(\ell)}(H) \mapsto L_W := P_W \xi P_W^{-1}$
- L_W solution of the KP-hierarchy
- $P_W = 1 + \sum_{i \geq 1} p_i \xi^{-i}$
- Each p_i rational expression in τ_W and its derivatives.

Solutions strict KP

- Take $W \in Gr^{(\ell)}(H)$ and $w_0 \in W, w_0 \neq 0$
- Let $w_0^{\perp} = \{w \mid w \in W, w \perp w_0\}$
- $(W, w_0) \mapsto M_{W,w_0} := Q_{W,w_0} \xi Q_{W,w_0}^{-1}$
- M_{W,w_0} solution of the strict KP-hierarchy
- $Q_{W,w_0} = \sum_{i \geq 0} q_i \xi^{-i}, q_0 \in R^*$
- Each q_i rational expression in τ_W , $\tau_{w_0^{\perp}}$ and their derivatives.

Solutions $LT(R)_{\geq 0}$-hierarchy

- Consider flags $\mathcal{F} = \{W_\ell\}$:
$$\cdots W_{\ell+1} \subset W_\ell \subset W_{\ell-1} \cdots$$
with $W_\ell \in Gr^{(\ell)}(H)$.
- $\mathcal{F} \mapsto \mathcal{L}_{\mathcal{F}} = U_{\mathcal{F}} \Lambda U_{\mathcal{F}}^{-1}$
- $\mathcal{L}_{\mathcal{F}}$ solution of the $LT(R)_{\geq 0}$-hierarchy
- $U_{\mathcal{F}} \in U_-$

$$U_{\mathcal{F}} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & 1 & 0 & 0 & \ddots \\ \ddots & \ddots & 1 & 0 & \ddots \\ \ddots & \ddots & u_{\ell\ell-1} & 1 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Solutions $LT(R)_{>0}$-hierarchy

- Consider flags $\mathcal{F} = \{W_\ell\}$, $W_\ell \in Gr^{(\ell)}(H)$:

$$\cdots W_{\ell+1} \subset W_\ell \subset W_{\ell-1} \cdots$$

  and a basis $F = \{\underline{w}_\ell\}$ of $\oplus_{\ell \in \mathbb{Z}} W_\ell / W_{\ell+1}$
- $\mathcal{F} + F \mapsto \mathcal{N}_{\mathcal{F}F} = P_{\mathcal{F}F} \Lambda P_{\mathcal{F}F}^{-1}$
- $\mathcal{N}_{\mathcal{F}F}$ solution of the $LT(R)_{>0}$-hierarchy
- $P_{\mathcal{F}F} \in P_-$

$$P_{\mathcal{F}F} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & p_{\ell-2\ell-2} & 0 & 0 & \ddots \\ \ddots & p_{\ell-1\ell-2} & p_{\ell-1\ell-1} & 0 & \ddots \\ \ddots & \ddots & \ddots & p_{\ell\ell} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Solutions two dimensional Toda

- Consider $g \in Gl_{res}^{(0)}(H)$
- Take the commuting flows

$$\gamma(t,s) = \exp(\sum_{i\geq 1} t_i z^i)\exp(\sum_{j\geq 1} s_j z^{-j})$$

- Decompose

$$\gamma(t,s)g\gamma(t,s)^{-1} = \mathcal{U}_-^{-1}\mathcal{P}_+$$

  with $\mathcal{U}_- \in U_-$ and $\mathcal{P}_+ \in P_+$.
- Solutions two dimensional Toda:

$$\mathcal{L} = \mathcal{U}_-\Lambda\mathcal{U}_-^{-1} \text{ and } \mathcal{M} = \mathcal{P}_+\Lambda^{-1}\mathcal{P}_+^{-1}$$

THANK YOU FOR YOUR ATTENTION