Integrable Systems and Moduli Spaces

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Moscow, December 2011

Curve results with T.P. Northover.
Monopole results in collaboration with V.Z. Enolski.

Overview

$$\begin{array}{ccc}
\text{Equations} & \xrightarrow{\textit{Zero Curvature/Lax}} & \text{Spectral Curve } \mathcal{C} \subset \mathcal{S} \\
\uparrow & & \downarrow \\
\text{Reconstruction} & \longleftarrow & \text{Baker-Akhiezer Function}
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Examples

- BPS Monopoles
- Sigma Model reductions in AdS/CFT
- KP, KdV solitons
- Harmonic Maps
- SW Theory/Integrable Systems

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Goals: Spectral curves are transcendental

$\mathcal{L}^2$ trivial $\Longleftrightarrow 2U \in \Lambda$ trivial $\Longleftrightarrow \theta\left(2\mathbf{U} + \mathbf{C}|\tau\right) = 0$, $\quad \int_{\mathfrak{c}} v \in \overline{\mathbb{Q}}$

Implementation of symmetry to simplify

Spectral Curves: data

- Homology basis $\{\gamma_i\}_{i=1}^{2g} = \{\mathfrak{a}_i, \mathfrak{b}_i\}_{i=1}^{g}$
  - algorithm for branched covers of $\mathbb{P}^1$ (Tretkoff & Tretkoff)
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- Period Matrix $\tau = \mathcal{B}\mathcal{A}^{-1}$ where

$$\Pi := \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} = \begin{pmatrix} \oint_{\mathfrak{a}_i} du_j \\ \oint_{\mathfrak{b}_i} du_j \end{pmatrix}$$

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- normalized holomorphic differentials ω_i, $\oint_{\mathfrak{a}_i} \omega_j = \delta_{ij}$ $\oint_{\mathfrak{b}_i} \omega_j = \tau_{ij}$
- $\mathcal{C}$ often has an antiholomorphic involution/real structure
  - reality constrains the form of the period matrix.
  - there may be between 0 and $g+1$ ovals of fixed points of the antiholomorphic involution.
  - Imposing reality can be one of the hardest steps.

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$\theta\left(t\mathbf{U}+\mathbf{C}|\tau\right)$

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- $e\equiv\phi_Q\left(\sum_{i=1}^{g-1}P_i\right)+K_Q, \qquad \phi_Q(P):=\int_Q^P\omega$

$$\mathrm{mult}_e\,\theta=\mathrm{i}\left(\sum_{\mathrm{i}=1}^{\mathrm{g}-1}\mathrm{P_i}\right)=\dim \mathrm{H}^1(\mathcal{C},\mathcal{L}_{\sum_{\mathrm{i}=1}^{\mathrm{g}-1}\mathrm{P_i}})=\dim\mathrm{H}^0(\mathcal{C},\mathcal{L}_{\sum_{\mathrm{i}=1}^{\mathrm{g}-1}\mathrm{P_i}})$$

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- $-K_Q=\phi_*\left(\Delta-(g-1)Q\right)=\phi_Q\left(\Delta\right),$
  $\deg\Delta=g-1, \quad 2\Delta\equiv\mathcal{K}_{\mathcal{C}}$

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Harmonic Maps $T^2 \to S^3$ Hitchin gives a bijective correspondence between harmonic maps and hyperelliptic curves $\mathcal{C}$: $\eta^2 = f(\lambda)$ satisfying various constraints including (two) third class differentials whose *periods are all integers.*

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Harmonic Maps $T^2 \to S^3$
Closed geodesics on an ellipsoid Abenda, Fedorov: $\exists$ a nontrivial cycle $\mathfrak{c} = \sum_{i=1}^n m_i \mathfrak{a}_i$, $m_i \in \mathbb{Z}$ such that

$$\int_{\mathfrak{c}} \omega_1 = T, \qquad \int_{\mathfrak{c}} \omega_j = 0, \quad \omega_j = z^{j-1} dz/y \quad j = 2, \ldots, n,$$

$T > 0$ the period of the geodesic, $\mathfrak{a}_i$ chosen so that $\int_{\mathfrak{a}_i} \omega_j \in \mathbb{R}$.

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$$2\mathbf{U} \in \Lambda \Longleftrightarrow \mathbf{U} = \frac{1}{2\pi\imath}\left(\oint_{\mathfrak{b}_1}\gamma_\infty, \ldots, \oint_{\mathfrak{b}_g}\gamma_\infty\right)^T = \frac{1}{2}\mathbf{n} + \frac{1}{2}\tau\mathbf{m}$$

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$\iff$ (Bilinear relations) $\exists$ 1-cycle $\mathfrak{es} = \mathbf{n}\cdot\mathfrak{a} + \mathbf{m}\cdot\mathfrak{b}$ s.t.

$$(\mathbf{n}, \mathbf{m})\begin{pmatrix}\mathcal{A}\\ \mathcal{B}\end{pmatrix} = -2(0, \ldots, 0, 1), \qquad du_g = \frac{\eta^{n-2}}{\frac{\partial\mathcal{P}}{\partial\eta}}\,d\zeta,$$

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Trancendental constraint. How do we impose them?

Transcendence

Lindemann If $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$ are l.i. over $\mathbb{Q}$ then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent. If $0 \neq \alpha \in \overline{\mathbb{Q}}$ then $e^{\alpha} \notin \overline{\mathbb{Q}}$

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$E := E(\mathbb{K}) \qquad \mathbb{Q} \subseteq \mathbb{K} \subset \overline{\mathbb{Q}}$

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3, \quad g_{2,3} \in \mathbb{K},\ \Delta \neq 0$$

Schneider $\wp(\alpha)$ transcendental for $0 \neq \alpha \in \overline{\mathbb{Q}}$

Schneider $a\omega + b\eta$ transcendental for $a, b \in \overline{\mathbb{Q}}$, not both zero, ω a period and $\eta = 2\zeta(\omega/2)$. The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ has transcendental circumference for $a, b \in \overline{\mathbb{Q}}$

Schneider $j(z)$ transcendental for $z \in \overline{\mathbb{Q}}$ and $Im(z)$ not a quadratic irrational.

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Periods The periods of a meromorphic differential ξ are either zero or transcendental (χ rational, $(x_j, y_j) \in E(\mathbb{K})$, $a, b \in \mathbb{K}$)

$$\xi = \sum_j c_j \frac{y - y_j}{x - x_j} \frac{dx}{y} + a \frac{dx}{y} + b\, x \frac{dx}{y} + d\chi$$

Siegel (1932),Schneider (1937), Laurent (1980), Wüstholz (1984)

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Bertrand 1997 $\theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$ transcendental for $0 < |q| < 1$, $q \in \overline{\mathbb{Q}}$

Analytic Subgroup Theorem (Wüstholz)

G a comm. alg. group defined over number field $\mathbb{K}$

$\mathfrak{g}$ =Lie(G), $\mathfrak{b}$ a subalgebra of $\mathfrak{g}$

$B := \exp_G(\mathfrak{b} \otimes_{\mathbb{K}} \mathbb{C}) \leq G(\mathbb{C})$ an analytic subgroup defined over $\mathbb{K}$

B not necessarily closed

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Problem Determine $B(\overline{\mathbb{K}}) := B \cap G(\overline{\mathbb{K}})$
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Analytic Subgroup Theorem (Wüstholz) Let $B \subseteq G(\mathbb{C})$ be an analytic subgroup defined over $\mathbb{K}$. Then $B(\overline{\mathbb{K}}) \neq 0$ iff there exists a nontrivial algebraic subgroup $H \leq G$ defined over a number field such that $H(\mathbb{C}) \leq B$.
Theorem (Rosenlicht) Commutative algebraic groups G extensions of Abelian varieties A (eg Generalized Jacobians)

$$0 \longrightarrow G_a^r \times G_m^s \longrightarrow G \longrightarrow A \longrightarrow 0$$

Analytic Subgroup Theorem: Example 1.

$G = G_a \times G_m$ defined over $\mathbb{Q}$, $\quad G(\mathbb{C}) = \mathbb{C} \times \mathbb{C}^*$

$\mathfrak{g} = \mathbb{Q} \times \mathbb{Q}$, $\quad \mathfrak{g} \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C} \times \mathbb{C}$, $\quad \exp_G : (z, w) \to (z, e^w)$

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Assume that both $\alpha \neq 0$ and e^α are algebraic

Let $\Delta \subset \mathbb{C} \times \mathbb{C}$ be the diagonal.

Then $B := \exp_G(\Delta)$ is connected and has dimension 1.

Now $(\alpha, \alpha) \in \Delta$ and $(\alpha, e^\alpha) \in B(\overline{\mathbb{Q}})$ by assumption.

Since $(\alpha, e^\alpha) \neq (0, 1)$ then $B(\overline{\mathbb{Q}})$ nontrivial.

$\therefore \exists$ proper algebraic subgroup $H \leq G$ s.t. $H(\mathbb{C}) \leq B$.

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Since $(\alpha, e^\alpha) \neq (0, 1)$ then $B(\overline{\mathbb{Q}})$ nontrivial.
$\therefore \exists$ proper algebraic subgroup $H \leq G$ s.t. $H(\mathbb{C}) \leq B$.

$H(\mathbb{C})$ nontrivial $\Longrightarrow \dim_{\mathbb{C}} H(\mathbb{C}) \geq 1$
$H(\mathbb{C}) \leq B$, $\quad \dim_{\mathbb{C}} B = 1 \Longrightarrow H(\mathbb{C}) = B$
$\therefore B$ is an algebraic subgroup
But B is the graph of the exponential function e^z which is evidently not algebraic.
Contradiction.

Both $\alpha \neq 0$ and e^α cannot be algebraic

Analytic Subgroup Theorem: Example 2.

E the elliptic curve over $\mathbb{K}$ as before.
E is a commutative algebraic group.
$\mathrm{Lie}\, E(\mathbb{C}) \cong \mathbb{C}, \quad \exp_E : \mathbb{C} \to \mathbb{P}^2(\mathbb{C}), \quad z \to [\wp(z), \wp'(z), 1]$

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Suppose the period $\omega \neq 0$ is algebraic

$$\exp_G(\omega/2, \omega/2) = (\omega/2, [\wp(\omega/2), \wp'(\omega/2), 1]) = (\omega/2, [\wp(\omega/2), 0, 1])$$

Now $\wp(\omega/2)$ satisfies $4x^3 - g_2 x - g_3 = 0$ so is algebraic.
Therefore $(\omega/2, [\wp(\omega/2), 0, 1]) \in B(\overline{\mathbb{K}})$ and so $B(\overline{\mathbb{K}})$ is nontrivial.

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As previously, $H(\mathbb{C}) = B$ and so B is an algebraic subgroup
Then $\wp(z)$ is algebraic, a contradiction (as infinitely many poles)
ω cannot be algebraic

Analytic Subgroup Theorem: Applications

X a quasiprojective variety/number field $\mathbb{K}$ with a $\mathbb{K}$-rational point.
$\omega \in H^0(X, \Omega^1_{X/\mathbb{K}})$ a holomorphic differential on X.
Faltings-Wüstholz: $(X, \omega) \to$ comm. alg. group over $\mathbb{K}$
Using the analytic subgroup theorem W. deduces that

Theorem $\int_\gamma \omega \quad (\gamma \in H_1(X, \mathbb{Z}))$ are either zero or transcendental.

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- Periods of elliptic curves are transcendental
- For $a, b \in \mathbb{Q}$, $a + b \notin \mathbb{Z}$ (Schneider 1948)

$$B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx \notin \overline{\mathbb{Q}}$$

Periods of Fermat curves.

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Holomorphic differential$\to$ Jacobian, 2nd kind differential$\to$ G_a,
3rd kind differential$\to$ G_m.

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Corollary $\pi \times {}_2F_1\left(\frac{1}{3}, \frac{1}{3}; 1; t\right) \notin \overline{\mathbb{Q}}$ for $t \in \mathbb{Q}$
First transcendental constraint: Number Theory+Ramanujan

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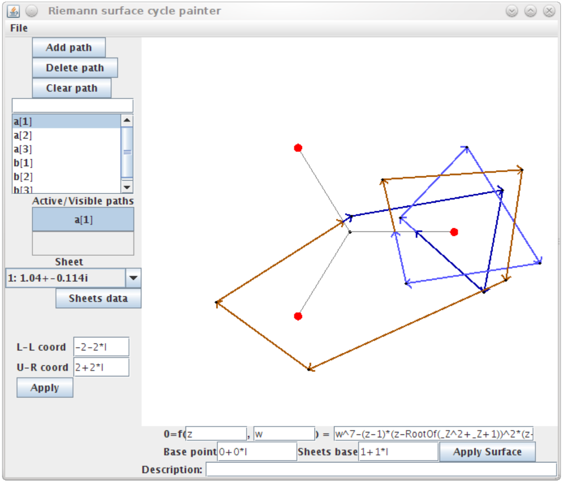

Riemann surface cycle painter
File
Add path
Delete path
Clear path
a[1]
a[2]
a[3]
b[1]
b[2]
b[3]
Active/Visible paths
a[1]
Sheet
1: 1.04+-0.114i
Sheets data
L-L coord
-2-2*I
U-R coord
2+2*I
Apply
0=f(
z
w
) =
w^7-(z-1)*(z-RootOf(_Z^2+_Z+1))^2*(z-
Base point
0+0*I
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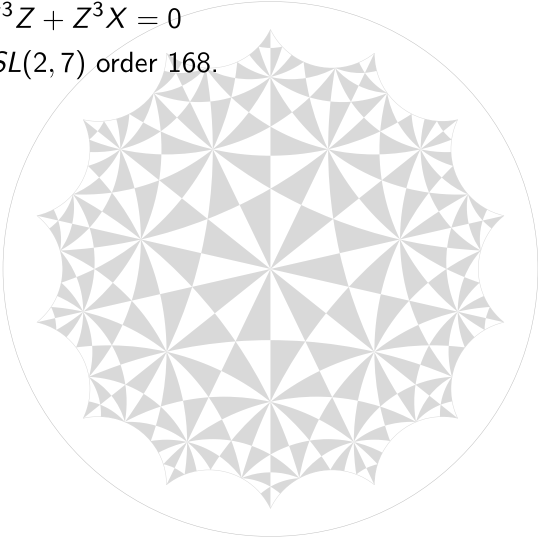
- How can one specify homology cycles?
- How to determine M, $\sigma_*(\gamma) = M.\gamma$? `extcurves`
- How to determine a good basis $\{\gamma_i\}$?

Example: Klein's Curve

- $\mathcal{C}$: $X^3Y + Y^3Z + Z^3X = 0$
- $\mathrm{Aut}(\mathcal{C}) = PSL(2,7)$ order 168.



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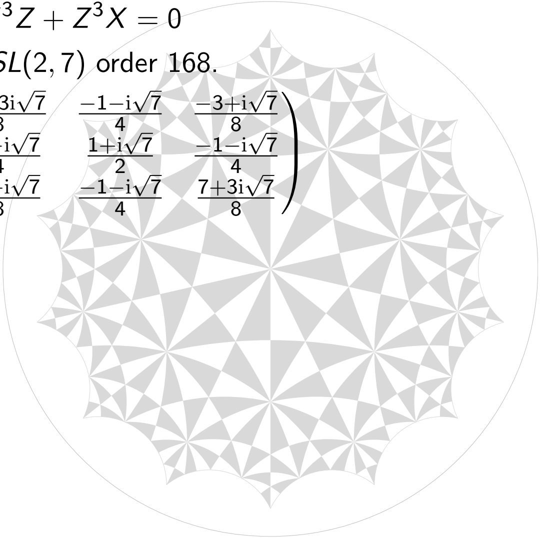

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- $\mathcal{C}$: $X^3Y + Y^3Z + Z^3X = 0$
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- $\mathcal{C}$: $w^7 = (z-1)(z-\rho)^2(z-\rho^2)^4, \quad \rho = \exp(2\pi \mathrm{i}/3)$

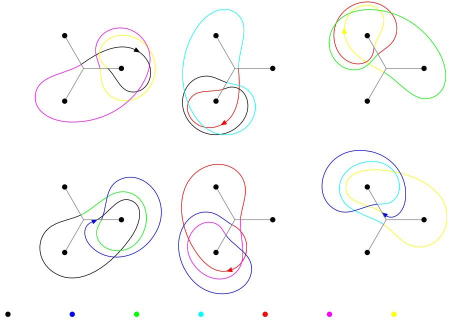

Figure: Homology basis in (z,w) coordinates

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$$-2K_Q = \phi_*\left(2\Delta - 2(g-1)Q\right) = \int_*^{2\Delta}\omega - 2(g-1)\int_*^{Q}\omega$$

$$-2K_Q.L = \int_*^{2\Delta}\sigma^*\omega - 2(g-1)\int_*^{Q}\sigma^*\omega$$

$$-2K_Q.\left[L-1\right] = \int_{2\Delta}^{\sigma(2\Delta)}\omega - 2(g-1)\int_{Q}^{\sigma(Q)}\omega$$

Lemma: $\sigma^N = \mathrm{Id}$. If $L-1$ is invertible and Q a fixed point of σ then K_Q is a $2N$-torsion point.

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Idea: Use Smith Normal Form

$$K_Q = \frac{i}{\sqrt{7}}(3,-1,5) \qquad Q = (z,w) = (\rho, 0)$$

How to determine a good basis $\{\gamma_i\}$?

Example: (Fay) $\phi : \hat{\mathcal{C}} \to \hat{\mathcal{C}},\ \phi^2 = \mathrm{Id},\quad \pi : \hat{\mathcal{C}} \to \mathcal{C} := \hat{\mathcal{C}}/ < \phi >$
$2n$ fixed points. $\hat{g} = 2g + n - 1$

$$\mathfrak{a}_1, \mathfrak{b}_1, \ldots \mathfrak{a}_g, \mathfrak{b}_g, \mathfrak{a}_{g+1}, \mathfrak{b}_{g+1}, \ldots \mathfrak{a}_{g+n+1}, \mathfrak{b}_{g+n+1}, \mathfrak{a}_{1'}, \mathfrak{b}_{1'}, \ldots \mathfrak{a}_{g'}, \mathfrak{b}_{g'}$$

where $\mathfrak{a}_{1'}, \mathfrak{b}_{1'}, \ldots, \mathfrak{a}_{g'}, \mathfrak{b}_{g'}$ a basis of $H_1(\mathcal{C}, \mathbb{Z})$ and

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$$\hat{\tau} = \begin{pmatrix} \dfrac{\pi+\tau}{2} & \Pi & \dfrac{\pi-\tau}{2} \\ p & 2P & p \\ \dfrac{\pi-\tau}{2} & \Pi & \dfrac{\pi+\tau}{2} \end{pmatrix} \qquad \begin{pmatrix} \pi & p \\ p & P \end{pmatrix} \in \mathfrak{H}_{g+n-1}$$

How to determine a good basis $\{\gamma_i\}$?

Theorem: Let $S \in Sp(2g, \mathbb{Z})$ be a symplectic involution, $S^T JS = J$ and $S^2 = \mathrm{Id}$. Then S is symplectically equivalent to one of the form $S = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ where

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Corollary: Let S be a symplectic involution of $W = \mathbb{Z}^{2g}$ with canonical pairing. Then $W = L_1 \oplus L_2$, $< L_i, L_i >= 0$, with stable Lagrangian subspaces $SL_i = L_i$.