The Gould-Hopper Polynomials in the Novikov-Veselov equation

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The Novikov-Veselov [1984] equation is defined by ($U$ and $t$ is real) [Athorne, Dubrovsu, Matveev, Nimmo, Salle, Y.Ohta]

$$U_t = \partial_z^3 U + \bar{\partial}_z^3 U - 3\partial_z (VU) - 3\bar{\partial}_z (\bar{V}U),$$

(1)

$$\bar{\partial}_z V = \partial_z U.$$ 

When $z = \bar{z} = x$, we get the KdV equation ($U = V = \bar{V}$)

$$U_t = 2U_{xxx} - 12UU_x.$$
The Novikov-Veselov [1984] equation is defined by ($U$ and $t$ is real) [Athorne, Dubrovsky, Matveev, Nimmo, Salle, Y.Ohta]

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The Novikov-Veselov equation can be represented as the form of Manakov’s triad

\[ H_t = [A, H] - BH, \]

where $H$ is the two-dimension Schrödinger operator

\[ H = \partial_z \overline{\partial}_z + U \]

and

\[ A = \partial_z^3 - V\partial_z + \overline{\partial}_z^3 - \overline{V}\overline{\partial}_z, \quad B = V\overline{z} + \overline{V}\overline{z}. \]
It is equivalent to the linear representation

\[ H\psi = 0, \quad \partial_t \psi = A\psi. \]  

(2)
Motives:

- Integrable deformation of Schrodinger Operator [Athorne, Matveev, Nimmo, Y.Ohta]
- D-bar dressing method [Boiti, Dubrovsky, Leon, Pempinelli, Tsai]
- Two dimensional generalization of KdV equation [S.P.Novikov, A.Veselov, L.V. Bogdanov, P.G. Grinevich]
- Two-component BKP equation [R. Hirota, I.Krichever, Takasaki, Si-Qi Liu, C.Z. Wu, Y.J. Zhang]
- The Tzitzeica equation [E.Ferapontov., A.E. Mironov]
- Integrable deformation of minimal Lagrangian tori in $CP^2$ [A.E. Mironov]
- Integrable deformation of Dirichlet-to-Neumann map (Electrical Impedance Tomography, [M.Lassas, J. Mueller, A.Stahel])
Let $H\psi = H\omega = 0$. Then via the Moutard transformation[1878]

$$U(z, \bar{z}) \rightarrow \hat{U}(z, \bar{z}) = U(z, \bar{z}) + 2\partial\bar{\partial}\ln[i \int (\psi\partial\omega \quad - \omega\partial\psi)dz - (\psi\bar{\partial}\omega - \omega\bar{\partial}\psi)d\bar{z}],$$

$$\psi \rightarrow \theta = \frac{i}{\omega} \int (\psi\partial\omega - \omega\partial\psi)dz - (\psi\bar{\partial}\omega - \omega\bar{\partial}\psi)d\bar{z}$$

one can construct a new Schrodinger operator

$$\hat{H} = \partial_z\bar{\partial}_z + \hat{U}$$

and $\hat{\psi} = \frac{1}{\theta}$ such that

$$\hat{H}\hat{\psi} = 0.$$
The extended Moutard transformation was established such that $\hat{U}(t, z, \bar{z})$ and $\hat{V}(t, z, \bar{z})$ defined by [Matveev, Salle, Athorne, Nimmo, H.C.Hu, S.Y.Lou and Q.P.Liu, 1991-2003]

$$\hat{U}(t, z, \bar{z}) = U(t, z, \bar{z}) + 2 \partial \bar{\partial} \ln iW,$$

where

$$W = \int (\psi \partial \omega - \omega \partial \psi) dz - (\psi \bar{\partial} \omega - \omega \bar{\partial} \psi) d\bar{z}$$

$$+ [\psi \partial^3 \omega - \omega \partial^3 \psi + \omega \bar{\partial}^3 - \psi \bar{\partial}^3 \omega$$

$$+ 2(\partial^2 \psi \partial \omega - \partial \psi \partial^2 \omega) - 2(\bar{\partial}^2 \psi \bar{\partial} \omega - \bar{\partial} \psi \bar{\partial}^2 \omega)$$

$$+ 3V(\psi \partial \omega - \omega \partial \psi) - 3\bar{V}(\psi \bar{\partial} \omega - \omega \bar{\partial} \psi)] dt, \quad (4)$$

$$\hat{V} = V + 2 \partial \bar{\partial} \ln iW, \quad (5)$$

will also satisfy the Novikov-Veselov equation.
In particular, we can use $U = V = 0$ as the seed solution. Then $H = \partial \bar{\partial}$. Let us consider the holomorphic functions $P(z, t)$:

$$
\frac{\partial P}{\partial t} = \frac{\partial^3 P}{\partial z}.
$$

(6)

Then we have the following
Theorem [Taimanov and Tsarev, 2008]
Let $\mathcal{P}_1(t,z)$ and $\mathcal{P}_2(t,z)$ be polynomial functions of $z$ and satisfy (6). One defines $\omega_1 = \mathcal{P}_1 + \bar{\mathcal{P}}_1$ and $\omega_2 = \mathcal{P}_2 + \bar{\mathcal{P}}_2$. Then

$$U(t, z, \bar{z}) = 2\partial\bar{\partial}\ln iW,$$

where

$$W = \mathcal{P}_1\bar{\mathcal{P}}_1 - \mathcal{P}_2\bar{\mathcal{P}}_2 + \int [ (\mathcal{P}_1'\mathcal{P}_2 - \mathcal{P}_1\mathcal{P}_2')dz 
+ (\bar{\mathcal{P}}_1\bar{\mathcal{P}}_2' - \bar{\mathcal{P}}_1'\bar{\mathcal{P}}_2)d\bar{z} 
+ \int [ \mathcal{P}_1'''\mathcal{P}_2 - \mathcal{P}_1\mathcal{P}_2''' + 2(\mathcal{P}_1'\mathcal{P}_2'' - \mathcal{P}_1''\mathcal{P}_2') + \bar{\mathcal{P}}_1\bar{\mathcal{P}}_2''' 
- \bar{\mathcal{P}}_1'''\bar{\mathcal{P}}_2 + 2(\bar{\mathcal{P}}_1''\bar{\mathcal{P}}_2' - \bar{\mathcal{P}}_1'\bar{\mathcal{P}}_2''')]dt,$$

$$V = 2\partial\bar{\partial}\ln iW,$$

is a solution of Novikov-Veselov equation, which is rational in $z, \bar{z}, t$. 
Taimanov and Tsarev considered the polynomial-type solutions for (6)

\[ P_N(t, z) = z^N + \sigma_1 z^{N-1} + \sigma_2 z^{N-2} + \cdots + \sigma_{N-1} z + \sigma_N. \]

Then the flow (6) generates the \(\sigma\)-flow:

\[ \dot{\sigma}_k = (N - k + 3)(N - k + 2)(N - k + 1)\sigma_k - 3, k = 1, 2, 3, \ldots, N. \]
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Then the flow (6) generates the \( \sigma \)-flow:

\[ \dot{\sigma}_k = (N - K + 3)(N - k + 2)(N - k + 1)\sigma_{k-3}, \quad k = 1, 2, 3 \cdots N. \]

It can be seen that \( \sigma_1, \sigma_2 \) are conserved quantities. Indeed, \( \sigma_1, \sigma_2, \cdots, \sigma_N \) are the elementary symmetric polynomials in the roots \( q_1, q_2, \cdots, q_N \) of \( P(z) \):

\[
\begin{align*}
\sigma_1(\vec{q}) &= -\sum_{i=1}^{N} q_i, \\
\sigma_2(\vec{q}) &= \sum_{i<j} q_i q_j, \\
\sigma_3(\vec{q}) &= -\sum_{i<j<k} q_i q_j q_k, \quad \cdots, \\
\sigma_N(\vec{q}) &= (-1)^{N} q_1 q_2 \cdots q_N. 
\end{align*}
\]
The integrable (even linear) evolution of $\vec{\sigma} = (\sigma_1, \sigma_2, \cdots, \sigma_N)$ induces a dynamical system on the symmetric product $S^N C$ of the complex roots. We call such a dynamical system on $S^N C$ a $\sigma$-system.
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Remark:
From (7), we see that given two solutions $P_1(t, z)$ and $P_2(t, z)$, by a substitution of $e^{i\lambda_1}P_1(t, z)$ and $e^{i\lambda_2}P_2(t, z)$, where $\lambda_1$ and $\lambda_2$ are real-valued constants, into (7) we obtain a solution of the Novikov-Veselov solution. Therefore, to each pair of holomorphic solutions of (6), we can get an $(S^1 \times S^1)$-family of solutions to the Novikov-Veselov equation.
Let’s write $P_N(t, z)$ as

$$P_N(t, z) = (z - q_1(t))(z - q_2(t)) \cdots (z - q_N(t)).$$

Then from the equation (6), one gets the root dynamics

$$\dot{q}_j = -6 \sum_{m<n, j\neq m,n}^N \frac{1}{(q_j - q_m)(q_j - q_n)}.$$  \hspace{1cm} (8)

For example, when $N=3$, we have

$$\dot{q}_1 = -6 \frac{1}{(q_1 - q_2)(q_1 - q_3)},$$

$$\dot{q}_2 = -6 \frac{1}{(q_2 - q_1)(q_2 - q_3)},$$

$$\dot{q}_3 = -6 \frac{1}{(q_3 - q_1)(q_3 - q_2)}.$$
For N=4, one has

\[ \dot{q}_1 = -6\left[ \frac{1}{(q_1 - q_2)(q_1 - q_3)} + \frac{1}{(q_1 - q_3)(q_1 - q_4)} + \frac{1}{(q_1 - q_2)(q_1 - q_4)} \right] \]

\[ \dot{q}_2 = -6\left[ \frac{1}{(q_2 - q_1)(q_2 - q_3)} + \frac{1}{(q_2 - q_3)(q_2 - q_4)} + \frac{1}{(q_2 - q_1)(q_2 - q_4)} \right] \]

\[ \dot{q}_3 = -6\left[ \frac{1}{(q_3 - q_1)(q_3 - q_2)} + \frac{1}{(q_3 - q_1)(q_3 - q_4)} + \frac{1}{(q_3 - q_2)(q_3 - q_4)} \right] \]

\[ \dot{q}_4 = -6\left[ \frac{1}{(q_4 - q_2)(q_4 - q_3)} + \frac{1}{(q_4 - q_1)(q_4 - q_2)} + \frac{1}{(q_4 - q_1)(q_1 - q_3)} \right] \]
For $N=4$, one has

\[
\begin{align*}
\dot{q}_1 &= -6\left[ \frac{1}{(q_1 - q_2)(q_1 - q_3)} + \frac{1}{(q_1 - q_3)(q_1 - q_4)} + \frac{1}{(q_1 - q_2)(q_1 - q_4)} \right], \\
\dot{q}_2 &= -6\left[ \frac{1}{(q_2 - q_1)(q_2 - q_3)} + \frac{1}{(q_2 - q_3)(q_2 - q_4)} + \frac{1}{(q_2 - q_1)(q_2 - q_4)} \right], \\
\dot{q}_3 &= -6\left[ \frac{1}{(q_3 - q_1)(q_3 - q_2)} + \frac{1}{(q_3 - q_1)(q_3 - q_4)} + \frac{1}{(q_3 - q_2)(q_3 - q_4)} \right], \\
\dot{q}_4 &= -6\left[ \frac{1}{(q_4 - q_2)(q_4 - q_3)} + \frac{1}{(q_4 - q_1)(q_4 - q_2)} + \frac{1}{(q_4 - q_1)(q_1 - q_3)} \right].
\end{align*}
\]

We notice that since $\sigma_1$ and $\sigma_2$ are conserved quantities, one knows that

\[
\sum_{i=1}^{N} q_i, \quad \sum_{i=1}^{N} q_i^2
\]

are conserved densities of (8)
The goal is to investigate the properties of the root dynamics (8):

- Initial Value Problem
- Lax Pair
- Asymptotic behavior
The generating function of the Gould-Hopper polynomials $P_N(t, z)$ is

\[ e^{\lambda z + \lambda^3 t} = \sum_{N=0}^{\infty} P_N(t, z) \frac{\lambda^N}{N!}. \]

Indeed, the Gould-Hopper polynomials $P_N(t, z)$ has the operator representation [1962]

\[ P_N(t, z) = e^{t\partial z^3} z^N = [1 + t\partial^3 z + \frac{t^2 \partial^6 z}{2!} + \frac{t^3 \partial^9 z}{3!} + \frac{t^4 \partial^{12} z}{4!} + \cdots ] z^N. \]

We remark that in general the Gould-Hopper polynomials are defined by $P_N^{(m)}(t, z) = e^{t\partial^m z} z^N$. Here we take $m = 3$. 
One notices that the Gould-Hopper polynomials $P_N(t, z)$ are characterized by (6) and $P_N(0, z) = z^N$. For example,

\begin{align*}
P_0 &= 1, \quad P_1 = z, \quad P_2 = z^2, \quad P_3 = z^3 + 6t, \quad P_4 = z^4 + 24tz, \\
P_5 &= z^5 + 60tz^2, \quad P_6 = z^6 + 120tz^3 + 360t^2 \\
P_7 &= z^7 + 210tz^4 + 2520z, \quad P_8 = z^8 + 336tz^5 + 10080t^2z^2 \\
P_9 &= z^9 + 504tz^6 + 30240t^2z^3 + 60480t^3 \\
P_{10} &= z^{10} + 720tz^7 + 75600t^2z^4 + 604800t^3z
\end{align*}

Actually, we have

\begin{align*}
P_N(t, z) &= \frac{[N/3]}{N!} \sum_{k=0}^{[N/3]} \frac{t^k z^{N-3k}}{k! (N-3k)!} \\
\frac{dP_N(t, z)}{dz} &= NP_{N-1}(t, z)
\end{align*}
From the operation calculus, one has

$$(z + 3t \partial_z^2) P_N(t, z) = P_{N+1}(t, z).$$

Hence we yield the recursive relation

$$P_{N+1}(t, z) = z P_N(t, z) + 3t(N - 1)(N - 2) P_{N-3}(t, z). \quad (10)$$
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\[P_{N+1}(t, z) = zP_N(t, z) + 3t(N - 1)(N - 2)P_{N-3}(t, z). \quad (10)\]

We notice that if we consider the equation (6) with the initial data of analytical function

\[P(0, z) = \sum_{N=0}^{\infty} \alpha_N z^N,\]

then the formal solution is

\[P(t, z) = e^{t\partial_z^3} \sum_{N=0}^{\infty} \alpha_N z^N = \sum_{N=0}^{\infty} \alpha_N P_N(t, z). \quad (11)\]
Remark:
The successive operations of the operator \((z + 3t \partial^2_z)\) on the solution (11) can help us construct more solutions of (6). For example, if \(P(0, z) = \sin z\), then we have,

\[
e^{t \partial^3_z} \sin z = e^{t \partial^3_z} \sum_{N=0}^{\infty} \frac{(-1)^N}{(2N + 1)!} z^{2N+1} = \sum_{N=0}^{\infty} \frac{(-1)^N}{(2N + 1)!} P_{2N+1}(t, z) = \sin(z - t).
\]

The last equation uses the fact \(e^{t \partial^3_z} e^{iz} = e^{iz+t}\). Hence

\[(z + 3t \partial^2_z)^N \sin(z - t), \quad N = 0, 1, 2, 3, 4, \ldots\]

are also solutions of (6).
Initial Value Problem: The root dynamics of \( \sigma \)-flow can be solved by

\[
H_N(t, z) = (z - q_1(t))(z - q_2(t)) \cdots (z - q_N(t))
= P_N(t, z) + C_1 P_{N-1}(t, z) + \cdots + C_N P_0(t, z),
\]

where the constants \( C_1, C_2, \cdots, C_{N-1}, C_N \) are determined by the initial values of \( q_1(0), q_2(0), \cdots, q_N(0) \), that is,

\[
C_1 = -\sum_{i=1}^{N} q_i(0), \quad C_2 = \sum_{i<j} q_i(0)q_j(0),
\]

\[
C_3 = -\sum_{i<j<k} q_i(0)q_j(0)q_k(0), \quad \cdots,
\]

\[
C_N = (-1)^N q_1(0)q_2(0) \cdots q_N(0).
\]

It can be seen that the solutions \( q_1(t), q_2(t), \cdots, q_N(t) \) can be obtained algebraically.
Lax pair:
Firstly, we study the root dynamics of the Gould-Hopper polynomials, which correspond to the initial values
\[ q_1(0) = q_2(0) = \cdots = q_N(0) = 0. \]
Let’s define the \( N \times N \) matrix by
\[
X(t) = \begin{cases} 
  a_{i,i+1} = 1, & \text{if } i = 1, 2, 3, \cdots, N; \\
  a_{i,i-2} = -3t(i - 1)(i - 2), & \text{if } i = 3, 4, \cdots, N - 1; \\
  0, & \text{otherwise.} 
\end{cases} 
\]
Then from the recursive relation (10), one knows that
\[
P_N(t, z) = \det(zI_N - X(t)).
\]
For example, when \( N = 3 \),
\[
X(t) = \begin{pmatrix}
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  -6t & 0 & 0 \\
\end{pmatrix};
\]
N=4,

\[ X(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -6t & 0 & 0 & 1 \\ 0 & -18t & 0 & 0 \end{pmatrix}; \]

N=5,

\[ X(t) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -6t & 0 & 0 & 1 & 0 \\ 0 & -18t & 0 & 0 & 1 \\ 0 & 0 & -36t & 0 & 0 \end{pmatrix}. \]
We can write $X(t)$ as

$$X(t) = R(t)QR^{-1}(t),$$

where $Q = \text{diag}(q_1(t), q_2(t), \cdots, q_N(t))$ and

$$R(t) = \begin{pmatrix}
P_0(q_1,t) & P_0(q_2,t) & P_0(q_3,t) & \cdots & P_0(q_N,t) \\
P_1(q_1,t) & P_1(q_2,t) & P_1(q_3,t) & \cdots & P_1(q_N,t) \\
P_2(q_1,t) & P_2(q_2,t) & P_2(q_3,t) & \cdots & P_2(q_N,t) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
P_N(q_1,t) & P_N(q_2,t) & P_N(q_3,t) & \cdots & P_N(q_N,t)
\end{pmatrix}.$$

(13)
For instance, when $N = 3$,

$$R(t) = \begin{pmatrix} 1 & 1 & 1 \\ q_1 & q_2 & q_3 \\ q_1^2 & q_2^2 & q_3^2 \end{pmatrix};$$

$N=4$,

$$R(t) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ q_1 & q_2 & q_3 & q_4 \\ q_1^2 & q_2^2 & q_3^2 & q_4^2 \\ q_1^3 + 6t & q_2^3 + 6t & q_3^3 + 6t & q_4^3 + 6t \end{pmatrix};$$

$N=5$,

$$R(t) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ q_1 & q_2 & q_3 & q_4 & q_5 \\ q_1^2 & q_2^2 & q_3^2 & q_4^2 & q_5^2 \\ q_1^3 + 6t & q_2^3 + 6t & q_3^3 + 6t & q_4^3 + 6t & q_5^3 + 6t \\ q_1^4 + 24t & q_2^4 + 24t & q_3^4 + 24t & q_4^4 + 24t & q_5^4 + 24t \end{pmatrix}.$$
From the initial value problem, we notice that the polynomials $t^n$ can be replaced by the elementary symmetric polynomials of the roots $q_1, q_2, \cdots, q_N$. Hence one has $R(q)$. It can be seen that
\[ \dot{X}(t) = RLR^{-1}, \]
where
\[ L = \dot{Q} + [M, Q], \quad M = R^{-1} \dot{R}. \]
For example, when $N=3$,

$$L(t) = \begin{pmatrix}
\dot{q}_1 & \dot{q}_2 & \dot{q}_3 \\
q_1 - q_3 & q_3 - q_1 & q_3 - q_2 \\
q_1 - q_2 & q_2 - q_1 & q_3 - q_1 \\
q_2 - q_3 & q_3 - q_1 & \dot{q}_1 \\
\end{pmatrix};$$

$N=4$,

$$L(t) = \begin{pmatrix}
\dot{q}_1 & \frac{q_2(q_2 - q_3)(q_2 - q_4) + 6}{(q_1 - q_3)(q_1 - q_4)} & \frac{q_3(q_3 - q_2)(q_3 - q_4) + 6}{(q_1 - q_2)(q_1 - q_4)} & \frac{q_4(q_4 - q_2)(q_4 - q_3) + 6}{(q_1 - q_2)(q_1 - q_3)} \\
\frac{q_1(q_1 - q_3)(q_1 - q_4) + 6}{(q_2 - q_3)(q_2 - q_4)} & \dot{q}_2 & \frac{q_3(q_3 - q_1)(q_3 - q_4) + 6}{(q_2 - q_4)(q_2 - q_1)} & \frac{q_4(q_4 - q_1)(q_4 - q_3) + 6}{(q_2 - q_3)(q_2 - q_1)} \\
\frac{q_1(q_1 - q_2)(q_1 - q_4) + 6}{(q_2 - q_3)(q_4 - q_3)} & \frac{q_2(q_2 - q_4)(q_2 - q_1) + 6}{(q_4 - q_3)(q_1 - q_3)} & \dot{q}_3 & \frac{q_4(q_4 - q_1)(q_4 - q_2) + 6}{(q_2 - q_3)(q_1 - q_3)} \\
\frac{q_1(q_1 - q_2)(q_1 - q_3) + 6}{(q_4 - q_3)(q_4 - q_2)} & \frac{q_2(q_2 - q_3)(q_2 - q_1) + 6}{(q_4 - q_3)(q_4 - q_1)} & \frac{q_3(q_3 - q_1)(q_3 - q_2) + 6}{(q_4 - q_1)(q_4 - q_2)} & \dot{q}_4 \\
\end{pmatrix};$$
Since
\[ \frac{dX(t)}{dt} = \begin{cases} a_{i,i-2} = -3(i-1)(i-2), & \text{if } i = 3, 4 \ldots, N - 1; \\ 0, & \text{otherwise,} \end{cases} \]
we know \( \frac{dX(t)}{dt} \) is a nilpotent matrix and hence \( L \) is a nilpotent one, too. So
\[ \text{tr}(L^r) = \text{tr}\left[ \frac{dX(t)}{dt} \right]^r = 0, \quad r = 1, 2, 3, \ldots, \ldots \]
Actually, a simple calculation yields
\[ L^\left[ \frac{N}{2} \right]+1 = 0, \quad N \geq 3. \]
Since
\[
\frac{dX(t)}{dt} = \begin{cases} 
  a_{i,i-2} = -3(i - 1)(i - 2), & \text{if } i = 3, 4 \cdots , N - 1; \\
  0, & \text{otherwise},
\end{cases}
\]
we know \( \frac{dX(t)}{dt} \) is a nilpotent matrix and hence \( L \) is a nilpotent one, too. So
\[
tr(L^r) = tr\left[ \frac{dX(t)}{dt} \right]^r = 0, \quad r = 1, 2, 3, \ldots , \ldots
\]
Actually, a simple calculation yields
\[
L^{N/2+1} = 0, \quad N \geq 3.
\]
Now,
\[
\frac{d^2 X(t)}{dt^2} = 0
\]
will imply the Lax equation
\[
\frac{dL(t)}{dt} = [L, M]. \tag{14}
\]
For $N = 3, 4, 5$, we see that, by Maple software, $q_i$ satisfies the following Goldfish model [Calogero, 2001], a limiting case of the Ruijesenaars-Schneider system:

$$\ddot{q}_i = 2 \sum_{j \neq i} \frac{\dot{q}_i \dot{q}_j}{q_i - q_j}.$$

The reason is that the Gould-Hopper polynomials $P_N, N = 3, 4, 5$ are linear in $t$-variable.
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The reason is that the Gould-Hopper polynomials $P_N, N = 3, 4, 5$ are linear in $t$-variable. For $N = 6$, we have from the diagonal terms of the Lax equation (14)

$$\ddot{q}_i = 2 \sum_{j \neq i}^{6} \frac{\dot{q}_i \dot{q}_j}{q_i - q_j} + \sum_{j=1}^{6} (\text{some quadratic terms of } \bar{q})q_j + 720 \prod_{i \neq j}^{6} (q_i - q_j).$$
Remark:
For the Goldfish Model

\[ \ddot{q}_i = 2 \sum_{j \neq i} \frac{\dot{q}_i \dot{q}_j}{q_i - q_j}, \]

its initial value problem can be solved by the statement:
Remark:
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\[ \ddot{q}_i = 2 \sum_{j \neq i} \frac{\dot{q}_i \dot{q}_j}{q_i - q_j}, \]

its initial value problem can be solved by the statement:
\[ z = q_i(t), i = 1, 2, \cdots, N \] are the N roots of the equation

\[ \sum_{i=1}^{N} \frac{\dot{q}_i(0)}{z - q_i(0)} = \frac{1}{t}. \]

It can be seen that it is a polynomial in \( z \) with coefficients linear in \( t \). Then the special choices of initial datum can get the solutions of the root dynamics (8) for the cases \( N = 3, 4, 5 \).
Secondly, we consider the general case. Let’s define the 2D Appell polynomials $G_n(z,t)$ by means of the generating function [G.Bretti and P.Ricci, 2004]:

$$A(\lambda)e^{\lambda z + \lambda^3 t} = \sum_{n=0}^{\infty} G_n(z,t) \frac{\lambda^n}{n!}, \quad (15)$$

where

$$A(\lambda) = \sum_{k=0}^{N} \frac{\Gamma_k}{k!} \lambda^k,$$

$\Gamma_k$s being constants and $\Gamma_0 = 1$. Then one has the following formula, noting that $\binom{N}{h} = \binom{N}{N-h}$,

$$G_N = \sum_{h=0}^{N} \binom{N}{N-h} \Gamma_{N-h} P_h(z,t). \quad (16)$$

It’s easy to see that the polynomials $G_n(z,t)$ also satisfy the linear equation (6).
Then we have

\[ \Gamma_{N-h} = \frac{C_{N-h}}{N} \cdot \frac{1}{(N-h)} \].
Then we have

\[ \Gamma_{N-h} = \frac{C_{N-h}}{(N-h)_N}. \] (17)

Now, it’s suitable to introduce the coefficients of the Taylor expansion

\[ \frac{A'(\lambda)}{A(\lambda)} = \sum_{n=0}^{\infty} \frac{\alpha_n \lambda^n}{n!}. \]

It can be seen that the coefficients \( \alpha_n \) can be expressed by \( \Gamma_0, \Gamma_1, \cdots, \Gamma_{n+1} \) (or the initial values (17)). For examples,

\[
\begin{align*}
\alpha_0 &= \Gamma_1, \\
\alpha_1 &= \Gamma_2 - \Gamma_1^2, \\
\alpha_2 &= \Gamma_3 - 3\Gamma_1\Gamma_2 + 2\Gamma_1^3, \\
\alpha_3 &= \Gamma_4 + 12\Gamma_1^2\Gamma_2 - 4\Gamma_1\Gamma_3 - 3\Gamma_2^2 - 6\Gamma_1^4, \\
&\vdots
\end{align*}
\]
The recurrence relation for the 2D Appell polynomial \( G_N(z, t) \) can be written as follows:

\[
\begin{align*}
G_0(z, t) &= 1 \\
G_N(z, t) &= (z + \alpha_0)G_{N-1}(z, t) + 3t(N - 1)(N - 2)G_{N-3}(z, t) \\
&\quad + \sum_{k=0}^{N-2} \binom{N-1}{k} \alpha_{N-k-1} G_k(z, t).
\end{align*}
\] (18)

A simple calculation can yield

\[
\begin{align*}
G_0(z, t) &= 1, \quad G_1(z, t) = z + \alpha_0, \quad G_2(z, t) = (z + \alpha_0)^2 + \alpha_1 \\
G_3(z, t) &= (z + \alpha_0)^3 + 3\alpha_1(z + \alpha_0) + \alpha_2 + 6t \\
G_4(z, t) &= (z + \alpha_0)^4 + 6\alpha_1(z + \alpha_0)^2 + (4\alpha_2 + 24t)(z + \alpha_0) + \alpha_3 \\
&\quad + 3\alpha_1^2.
\end{align*}
\]

When \( A(\lambda) = 1 \), this recursive relation becomes (10). Hence the relation (18) is a generalization of (10) for arbitrary initial data.
The matrix corresponding to (12) can be constructed as follows:

\[ X(t) = \begin{cases} 
  a_{i,i+1} = 1, & \text{if } i = 1, 2, 3, \ldots, N - 1; \\
  a_{i,i} = -\alpha_0, & \text{if } i = 1, 2, 3, \ldots, N; \\
  a_{i,i-1} = -\binom{i-1}{i-2}\alpha_1, & \text{if } i = 2, 3, 4, \ldots, N; \\
  a_{i,i-2} = -\binom{i-1}{i-3}(6t + \alpha_2), & \text{if } i = 3, 4, 5, \ldots, N; \\
  a_{i,i-3} = -\binom{i-1}{i-4}\alpha_3, & \text{if } i = 4, 5, 6, \ldots, N; \\
  \vdots \\
  a_{i,i-k} = -\binom{i-1}{i-k-1}\alpha_k, & \text{if } i = k + 1, k + 2, k + 3, \ldots, N; \\
  \vdots \\
  a_{N,1} = -\alpha_{N-1} 
\end{cases} \]

Similarly, one has

\[ G_N(z, t) = \det(zI_N - X(t)). \]
For instance, when $N = 5$, we get

\[
X(t) = \begin{pmatrix}
-\alpha_0 & 1 & 0 & 0 & 0 \\
-\alpha_1 & -\alpha_0 & 1 & 0 & 0 \\
-(6t + \alpha_2) & -2\alpha_1 & -\alpha_0 & 1 & 0 \\
-\alpha_3 & -(18t + 3\alpha_2) & -3\alpha_1 & -\alpha_0 & 1 \\
-\alpha_4 & -4\alpha_3 & -(36t + 6\alpha_2) & -4\alpha_1 & -\alpha_0 \\
\end{pmatrix}.
\]

Also, one can write $X(t)$ as

\[
X(t) = R(t)Q R^{-1}(t),
\]

where $Q = \text{diag}(q_1(t), q_2(t), \ldots, q_N(t))$ and $R(t)$ is defined as (13) with $P_m(q_i, t)$ being replaced by $G_m(q_i, t)$. Then one follows the previous procedures and finally can get the Lax equation (14) for general case. Therefore the root dynamics (8) is Lax-integrable.

We notice here that for $N = 3, 4, 5$ the root dynamics of $G_N$ also satisfies the Gold-fish model.
Asymptotic behavior

It is known that the Gould-Hopper polynomial $P_N(t,z)$ has the scaling property:

$$P_N(t,z) = t^{\frac{N}{3}} \hat{P}_N\left(\frac{z}{t^{1/3}}\right), \quad (19)$$

where $\hat{P}_N(\eta)$ is the so-called generalized Hermite polynomial (or Appell polynomial) in $\eta = \frac{z}{t^{1/3}}$. For example,

$$P_8(t,z) = z^8 + 336tz^5 + 10080t^2 = t^{\frac{8}{3}} [\eta^8 + 336\eta^5 + 10080\eta^2] = t^{\frac{8}{3}} \hat{P}_8(\eta).$$

Then the k-th zero $\lambda_N^{(k)}$ (constant) of $\hat{P}_N(\eta)$ determines the dynamics of the root $q_k$, i.e.,

$$q_k(t) = \lambda_N^{(k)} t^{1/3}.$$
Since $\hat{P}_N(\xi\lambda_N^{(k)}) = 0, \xi^3 = 1$, one knows that the roots $q_k$ are located on the circles in the plane with time dependent radius or fixed at the origin.
Since $\hat{P}_N(\xi \lambda_N^{(k)}) = 0, \xi^3 = 1$, one knows that the roots $q_k$ are located on the circles in the plane with time dependent radius or fixed at the origin. Finally, from the Initial value Problem, we know that when $t \to \infty$ and $z \to \infty$ such that $|z|^3/t \to \text{constant}$, $P_N(t, z)$ plays the dominant role. Hence one yields

$$q_k(t) \to \lambda_N^{(k)} t^{1/3}.$$  

Consequently, the roots asymptotically will follow diagonal lines.
In this section, we establish smooth rational solutions for all time by the extended Moutard transformation (7) and Gould-Hopper polynomials.

Example 1
Let \( P_1 = z^2 + z + 1 \) and \( P_2 = -iz^2 - 2iz \). Then a simple calculation can yield the imaginary part of \( W \) in (7)

\[
M(x, y, t) = (x^2 + y^2)^2 + \frac{8}{3}x^3 + 4xy^2 + 4x^2 + 4x + 4t + 100.
\]

We can see that

\[
M(x, y, 0) \approx 4x + 100 \quad \text{near} \quad (0, 0)
\]

and

\[
M(x, y, 0) \approx (x^2 + y^2)^2 \quad \text{near} \quad r = \sqrt{x^2 + y^2} = \infty.
\]

It can be verified that \( M(x, y, 0) \) is positive for all \( \mathbb{R}^2 \). Also, \( M(x, y, t) \) is positive for all \( \mathbb{R}^2 \) at any fixed time \( t \). Then the solution \( U \) of the Novikov-Veselov equation (1) is
where

\[ M_1 = -12 \left( 294 + 600 x^2 + 588 y^2 + 8 x^3 + 888 x + 12 t + 3 x^4 
- 6 x^2 y^2 - 2 x^3 y^2 + 24 x^2 t - 3 y^4 x + 24 y^2 t + 36 x t - 3 y^4 + x^5 \right) \]

and

\[ M_2 = \left( 3 x^4 + 6 x^2 y^2 + 3 y^4 + 8 x^3 + 12 x y^2 + 12 x^2
+ 12 x + 12 t + 300 \right)^2 \]

At fixed time \( t \), one knows \( U \) decays like \( \frac{1}{r^3} \) for \( r \to \infty \). Also, \( U \) tends asymptotically to zero at the rate \( \frac{1}{t} \) at any fixed point \((x, y)\) when \( t \to \infty \).
Example 2
Let \( P_1 = (z^3 + 6t) + 2iz \) and \( P_2 = -i(z^3 + 6t) + z \). Then a simple calculation can yield the imaginary part of \( W \) in (7)

\[
f(x, y, t) = (x^2 + y^2)^3 + 4x^3y + 8xy^3 + 2(x^2 + y^2) \\
+ 6t(2x^3 - 6xy^2 - y) + 36t^2 + 6000.
\]
Example 2

Let $\mathcal{P}_1 = (z^3 + 6t) + 2iz$ and $\mathcal{P}_2 = -i(z^3 + 6t) + z$. Then a simple calculation can yield the imaginary part of $W$ in (7)

\[
  f(x, y, t) = (x^2 + y^2)^3 + 4x^3y + 8xy^3 + 2(x^2 + y^2) + 6t(2x^3 - 6xy^2 - y) + 36t^2 + 6000.
\]

Then we can see that

\[
  f(x, y, 0) \approx 2(x^2 + y^2) + 6000 \quad \text{near} \quad (0, 0)
\]

and

\[
  f(x, y, 0) \approx (x^2 + y^2)^3 \quad \text{near} \quad r = \sqrt{x^2 + y^2} = \infty.
\]

It can be verified that $f(x, y, 0)$ is positive for all $\mathbb{R}^2$.
Also, letting $f(x, y, t)$ be equal to zero, one has

\[
t = \frac{1}{2}xy^2 + \frac{1}{12}y - \frac{1}{6}x^3 \\
\pm \frac{1}{12}\sqrt{24x^2y^4 - 20xy^3 - 36x^4y^2 - 7y^2 - 20x^3y - 4y^6}{-8x^2 - 23996}.
\]

A simple calculation shows that the equation inside the square root is negative for all $\mathbb{R}^2$. Hence $f(x, y, t)$ is positive for all $\mathbb{R}^2$ at any fixed time $t$. Then a solution $U$ of the Novikov-Veselov equation (1) is

\[
U = \frac{F_1}{F_2},
\]

where
\[ F_1 = -\frac{1}{2} \left[ 24x^7y + 16x^6 + 24x^5y^3 + (432ty + 216000 + 48y^2)x^4 \
\right. \\
+ \left. (-24y^5 - 144t + 48y)x^3 + (432000y^2 + 864ty^3 - 96y^4)x^2 \
+ (-24y^7 - 48y^3 + 432ty^2 + (432000 + 1728t^2)y)x + 48000 \
+ 432y^5t - 32y^6 + 216000y^4 + 252t^2 \right] \\
\]

and

\[ F_2 = \left( x^6 + 3x^4y^2 + 3x^2y^4 + y^6 + 4x^3y + 8xy^3 + 2x^2 + 2y^2 \
+ 12tx^3 - 36txy^2 - 6ty + 36t^2 + 6000 \right)^2 \\
\]

At fixed time \( t \), one knows \( U \) decays like \( \frac{1}{r^4} \) for \( r \rightarrow \infty \). Also, \( U \) tends asymptotically to zero at the rate \( \frac{1}{t^2} \) at any fixed point \( (x, y) \) when \( t \rightarrow \infty \).
Example 3
Let $\mathcal{P}_1 = (z^5 + 60tz^2) + 2iz$ and $\mathcal{P}_2 = -i(z^5 + 60tz^2) + z$. Then the imaginary part of $W$ in (7) is

$$h(x, y, t) = \left( (x^2 + y^2)^5 - \frac{10}{3} x^5 y + \frac{20}{3} x^3 y^3 \right)$$
$$+ (x^2 + y^2) \left( 2 + 12 x^3 y - 12 xy^3 \right)$$
$$+ 120 t \left( x^2 + y^2 \right) \left( x^5 - 3 y^4 x - 2 x^3 y^2 \right) + 3600 t^2 \left( x^2 + y^2 \right)^2$$
$$+ 20 t \left( 11 y^3 + 3 x^2 y \right) + 120 t^2 + 1000 \right).$$

Similarly, $h(x, y, t)$ is positive for all $\mathbb{R}^2$ at fixed time $t$. Then the solution $U$ is

$$U = \frac{H_1}{H_2},$$

where
\[ H_1(x, y, t) = -\frac{1}{2} [360 \, x y^{13} + (360 \, x^3 + 14400 \, t) \, y^{11} - 144 \, y^{10} + (14400 \, x^2 t - 1800 \, x^5) \, y^9 + (\text{higher-order terms in } x, y, t) ] \]

and

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The Gould-Hopper Polynomials in the Novikov-Veselov equation
\[ H_2(x, y, t) = (3x^{10} + 15x^8y^2 + 30x^6y^4 + 30y^6x^4 + 15y^8x^2 \]
\[ + 3y^{10} + 26x^5y + 20x^3y^3 \]
\[ + 6x^2 + 6y^2 - 36xy^5 + 360x^7t - 1800tx^3y^4 - 360tx^5y^2 - 1080txy^6 \]
\[ + 10800x^4t^2 + 21600t^2x^2y^2 + 10800t^2y^4 + 660ty^3 + 180tx^2y \]
\[ + 360t^2 + 3000)^2 \]

In this case, at fixed time, \( U \) decays like \( \frac{1}{r^6} \) for \( r \to \infty \); moreover, we see that

\[ U \to \frac{-240(x^2 + y^2)}{[30(x^2 + y^2)^2 + 1]^2} = \frac{-240z\bar{z}}{[30z^2\bar{z}^2 + 1]^2} \text{ as } t \to \infty \quad (20) \]

at fixed point \((x, y)\), which is a stationary solution of (1) for

\[ V = \frac{3600 \bar{z}^4\bar{z}^2 - 120 \bar{z}^2}{(30\bar{z}^2\bar{z}^2 + 1)^2}. \]
We notice that if we define

\[ u(z, \bar{z}) = \ln U \quad \text{and} \quad V(z, \bar{z}) = \frac{U_{zz}}{3U}, \]

then the stationary equation of the Novikov-Veselov equation (1) will become the Tzitzeica equation [B.C. Konopelchenko and U. Pinkall., 1998]

\[ u_{z\bar{z}} = e^u + e^{-2u}, \]

where \( \epsilon \) is an arbitrary constant. It can be verified that (20) satisfies the 0 case, i.e., the Liouville equation

\[ \ln \frac{-2\kappa}{dS}{dz}^2 (1 + \kappa |S|^2)^2, \]

where \( S(z) \) is a locally univalent meromorphic function in some domain and \( \kappa \) is a constant. For the solution corresponding to (20), one knows that \( S(z) = z^2 \) and \( \kappa = 30 \).
We notice that if we define

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\[ u_{z\bar{z}} = e^u + \epsilon e^{-2u}, \quad (21) \]

where \( \epsilon \) is an arbitrary constant. It can be verified that (20) satisfies the \( \epsilon = 0 \) case, i.e., the Liouville equation, whose real solutions are given by

\[ \ln \frac{-2\kappa |\frac{dS}{dz}|^2}{(1 + \kappa |S|^2)^2}, \quad (22) \]

where \( S(z) \) is a locally univalent meromorphic function in some domain and \( \kappa \) is a constant. For the solution corresponding to (20), one knows that \( S(z) = z^2 \) and \( \kappa = 30 \).
For general case, let’s choose

\[ \mathcal{P}_1 = P_N(t, z) + 2iz \quad \text{and} \quad \mathcal{P}_2 = -iP_N(t, z) + z. \]

We can expect the imaginary part of \( W \) in (7) is positive for all \( \mathbb{R}^2 \) at any fixed time if we choose an appropriate constant. And the solution \( U(x, y, t) \) at any fixed time decays like \( \frac{1}{r^{N+1}} \) for \( N \geq 2 \). When \( t \to \infty \) and then one obtains the solutions (22) of the Liouville equation or the ones of the Tzitzeica equation (21).
For general case, let’s choose

\[ P_1 = P_N(t, z) + 2iz \quad \text{and} \quad P_2 = -iP_N(t, z) + z. \]

We can expect the imaginary part of \( W \) in (7) is positive for all \( \mathbb{R}^2 \) at any fixed time \textbf{if we choose an appropriate constant}. And the solution \( U(x, y, t) \) at any fixed time \textbf{decays like} \( \frac{1}{r^{N+1}} \) for \( N \geq 2 \). When \( t \to \infty \) and then one obtains the solutions (22) of the Liouville equation or the ones of the Tzitzeica equation (21).\textbf{Remark:} For each potential \( U(x, y, t) \) there exist \textbf{infinitely many} wave functions, which can be constructed by the Pfaffian of linear combinations of the Gould-Hopper polynomials.
In summary, one investigates

- The Gould-Hopper polynomials
- The $\sigma$-flow and the Lax pair
- The Gold-Fish model
- The smooth rational solutions of NV equation
- The solution of the Liouville equation (or the Tzitzeica equation)
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Thanks for your attention.