Asymptotic analysis of autoresonance in the system with small dissipation

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Outline

1. Statement of problem
2. Examples of model systems
3. Autoresonance
4. Numerical simulation
5. Asymptotic analysis
6. Effect of dissipation
7. Art of asymptotics
8. Art of stability
Statement of the problem

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Object

The solution with increasing amplitude \( \rho(t) \to \infty \) as \( t \to \infty \), which is stable.
Initial stage of autoresonance. Zero dissipation $\beta = 0$.

Constant driver amplitude: $f_1 = 0$. Autoresonance solution.
## Model systems. First example.

### Perturbed pendulum

\[
\frac{d\rho}{dt} = \sin \psi, \quad \frac{d\psi}{dt} = \rho - \lambda t, \quad \lambda = \text{const} \neq 0.
\]
Model systems. First example.

**Perturbed pendulum**

\[ \frac{d \rho}{dt} = \sin \psi, \quad \frac{d \psi}{dt} = \rho - \lambda t, \quad \lambda = \text{const} \neq 0. \]

\[ \frac{d^2 \psi}{dt^2} = \sin \psi - \lambda. \]
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**Phase portrait on the \((\psi, \dot{\psi})\) plain under different \(\lambda\)**
Model systems. First example.

Perturbed dissipationless pendulum

\[ \frac{d\rho}{dt} = \sin \psi, \quad \frac{d\psi}{dt} = \rho - \lambda t. \]

Dissipation effect

\[ \frac{d\rho}{dt} = \sin \psi - \gamma \rho, \quad \frac{d\psi}{dt} = \rho - \lambda t. \]
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\[ \lambda, \gamma = \text{const} > 0 \]
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\]

\(\lambda, \gamma = \text{const} > 0\)

Growing pumping & dissipation effect

\[
\frac{d\rho}{dt} = (f_0 + f_1 t) \sin \psi - \gamma \rho, \quad \frac{d\psi}{dt} = \rho - \lambda t, \quad (f_0, f_1 = \text{const}).
\]
Perturbed pendulum

\[ \frac{d\rho}{dt} = \sin \psi, \quad \frac{d\psi}{dt} = \rho - \lambda t. \]

Main autoresonance equations

\[ \frac{d\rho}{dt} = \sin \psi, \quad \rho \left[ \frac{d\psi}{dt} - \rho^2 + \lambda t \right] = b \cos \psi, \quad (b = \text{const}). \]
Model systems. Second example.

Perturbed pendulum

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\[ \frac{d\rho}{dt} = (f_0 + f_1 t) \sin \psi - \gamma \rho, \quad \rho \left[ \frac{d\psi}{dt} - \rho^2 + \lambda t \right] = b \cos \psi. \]
What is the autoresonance
What is the autoresonance

\[ \frac{d^2 x}{dt^2} + \sin x = 0 \]

Phase portrait of the unperturbed pendulum

**Figure:** Pendulum trajectories on the \((x, \dot{x})\) plane
Problem of the perturbed pendulum dynamic

Is it possible to reach large amplitude oscillation $x \approx 1$ starting near equilibrium $x = \dot{x} = 0$ and using a weak pumping $0 < \varepsilon \ll 1$ under small dissipation $0 < \Gamma \ll 1$?

Naive resonance perturbation of the pendulum:

$$\frac{d^2 x}{dt^2} + \sin x = -\Gamma \frac{dx}{dt} + \varepsilon \cos t.$$
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- The unperturbed pendulum is not isochronous.
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- This property prevents the growing of the amplitude under constant driver frequency.
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Advanced perturbation of the pendulum
\[ \frac{d^2x}{dt^2} + \sin x = -\Gamma \frac{dx}{dt} + \varepsilon (1 + \delta t) \cos(t - \alpha t^2). \]
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\[ \frac{d^2 x}{dt^2} + \sin x = -\Gamma \frac{dx}{dt} + \varepsilon (1 + \delta t) \cos(t - \alpha t^2). \]

Perturbed equation is not integrable.
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- The unperturbed pendulum is not isochronous.
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Perturbed equation is not integrable.

Solution is analyzed by either numerical simulation or asymptotic methods.
Pendulum under oscillating pumping. Simulation.

\[
\frac{d^2 x}{dt^2} + \sin x = -\Gamma \frac{dx}{dt} + \varepsilon (1 + \delta t) \cos (t - \alpha t^2).
\]

Small parameters:
\[
\varepsilon = 10^{-4}, \quad \Gamma = 0 \div 10^{-4}, \quad \delta = 0 \div 10^{-4}, \quad \alpha = 0 \div 3 \cdot 10^{-6}.
\]

Input: Small initial data
\[
[ \dot{x}^2 + x^2 ]_{t=0} \leq \varepsilon^2 / 3.
\]

Output: Energy
\[
E = \frac{1}{2} \dot{x}^2(t) + 1 - \cos x(t), \quad 0 < t \leq O(\varepsilon^{-1}).
\]
Pendulum under oscillating pumping. Simulation.

Equation

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\frac{d^2 x}{dt^2} + \sin x = -\Gamma \frac{dx}{dt} + \varepsilon (1 + \delta t) \cos(t - \alpha t^2).
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Pendulum under oscillating pumping. Simulation.

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\varepsilon = 10^{-4}, \Gamma = 0 \div 10^{-4}, \delta = 0 \div 10^{-4}, \alpha = 0 \div 3 \cdot 10^{-6}.
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Nonlinear resonance – driver frequency is constant: \( \alpha = 0 \)

**Constant driver amplitude.**

\[
\frac{d^2 x}{dt^2} + \sin x = -\Gamma \frac{dx}{dt} + \varepsilon \cos t.
\]
Nonlinear resonance – driver frequency is constant: \( \alpha = 0 \)

Constant driver amplitude. Zero dissipation \( \Gamma = 0 \)
Nonlinear resonance – driver frequency is constant: $\alpha = 0$

Constant driver amplitude. Effect of dissipation $\Gamma = \varepsilon$. 
Nonlinear resonance – driver frequency is constant: $\alpha = 0$

Constant driver amplitude. Adiabatic approximation.
Nonlinear resonance – driver frequency is constant: $\alpha = 0$

**Increasing driver amplitude** $\varepsilon + \varepsilon^2 t$.

\[
\frac{d^2 x}{dt^2} + \sin x = -\Gamma \frac{dx}{dt} + \varepsilon (1 + \varepsilon t) \cos t.
\]
Nonlinear resonance – driver frequency is constant: $\alpha = 0$

Increasing driver amplitude $\varepsilon + \varepsilon^2 t$. Zero dissipation $\Gamma = 0$
Nonlinear resonance – driver frequency is constant: $\alpha = 0$

Increasing driver $\varepsilon + \varepsilon^2 t$. Effect of dissipation $\Gamma = \varepsilon$. 
Nonlinear resonance – driver frequency is constant: $\alpha = 0$

Increasing driver. Adiabatic approximation.
How to get large energy under weak pumping?
Conclusion from simulation.

It is impossible to get large energy by using a constant driver frequency.
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It is impossible to get large energy by using a constant driver frequency.

Idea is
- To vary the driver frequency in order to keep the resonance with the free frequency while the energy begins to grow.
How to get large energy under weak pumping?

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References
Simulation the autoresonance. Zero dissipation.

Pendulum under oscillating pumping

\[
\frac{d^2 x}{dt^2} + \sin x = \varepsilon \cos(t - \alpha t^2).
\]

Variation of the pumping frequency as \( \alpha > 0 \).
Dissipationless systems. Decreasing driver frequency: $1 - 6 \cdot 10^{-6} t$

Constant driver amplitude $\varepsilon = 10^{-4}$. Two type of solutions.
Dissipationless systems. Decreasing driver frequency: $1 - 6 \cdot 10^{-6} t$

Constant driver amplitude $\varepsilon = 10^{-4}$. Initial stage.
Dissipationless systems. Decreasing driver frequency: \( 1 - 6 \cdot 10^{-6} t \)

Constant driver amplitude \( \varepsilon = 10^{-4} \). Long times.
Dissipation systems.
Dissipation systems.

Is autoresonance possible in dissipation system?

\[ \frac{d^2 x}{dt^2} + \sin x = -\Gamma \frac{dx}{dt} + \varepsilon \cos(t - \alpha t^2), \quad \Gamma \approx \varepsilon > 0. \]
Dissipation systems. Decreasing driver frequency: $1 - 6 \cdot 10^{-6}t$

Constant driver amplitude. Effect of dissipation $\Gamma = \varepsilon$. 
How to get large energy in dissipation system under $\Gamma > 0$
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The result from simulation: Dissipation suppresses oscillations and all solutions vanish with time at infinity.
How to get large energy in dissipation system under $\Gamma > 0$

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Idea is

- To enlarge the driver amplitude in order to both compensate dissipation losses of energy and keep system in the autoresonant mode.
How to get large energy in dissipation system under $\Gamma > 0$

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- To enlarge the driver amplitude in order to both compensate dissipation losses of energy and keep system in the autoresonant mode.

\[
\frac{d^2 x}{dt^2} + \sin x = -\Gamma \frac{dx}{dt} + \varepsilon (1 + \delta t) \cos(t - \alpha t^2), \quad 0 < \Gamma, \varepsilon, \delta, \alpha \ll 1.
\]
How to get large energy in dissipation system under $\Gamma > 0$

The result from simulation: Dissipation suppresses oscillations and all solutions vanish with time at infinity.

**Idea is**

- To enlarge the driver amplitude in order to both compensate dissipation losses of energy and keep system in the autoresonant mode.

$$\frac{d^2x}{dt^2} + \sin x = -\Gamma \frac{dx}{dt} + \varepsilon(1 + \delta t) \cos(t - \alpha t^2), \; 0 < \Gamma, \varepsilon, \delta, \alpha \ll 1.$$ 

**References**

Decreasing driver frequency: \(1 - 6 \cdot 10^{-6} t\)

Increasing driver amplitude: \(\varepsilon + 10^{-5} t\). Zero dissipation.
Decreasing driver frequency: $1 - 6 \cdot 10^{-6} t$

Increasing driver amplitude: $\varepsilon + 10^{-4} t$. Dissipation effect.
The problem of separation

\[
d^2 x/dt^2 + \sin x = -\Gamma dx/dt + \epsilon (1 + \delta t) \cos(t - \alpha t^2),
\]
\(0 < \Gamma, \epsilon, \delta, \alpha \ll 1\).

Conclusion from simulations

The autoresonance phenomenon depends on the initial data.

Main problem in autoresonance theory

How to separate the different types of solutions?

Suggest: Analysis on the initial stage.
The problem of separation

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\frac{d^2 x}{dt^2} + \sin x = -\Gamma \frac{dx}{dt} + \varepsilon (1 + \delta t) \cos(t - \alpha t^2), \quad 0 < \Gamma, \varepsilon, \delta, \alpha \ll 1.
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The autoresonance phenomenon depends on the initial data.
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Suggest: Analysis on the initial stage.
Analytical results for pendulum
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An example of nonlinear oscillator under perturbation

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\frac{d^2 x}{dt^2} + \sin x = -\Gamma \frac{dx}{dt} + \varepsilon (1 + \delta t) \cos(t - \alpha t^2), \quad 0 < \varepsilon \ll 1.
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Analytical results for pendulum

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\[ \frac{d^2 x}{dt^2} + \sin x = -\Gamma \frac{dx}{dt} + \varepsilon (1 + \delta t) \cos(t - \alpha t^2), \quad 0 < \varepsilon \ll 1. \]

Anzatz. Asymptotic approximation on the initial stage

\[ x(t; \varepsilon) = \varepsilon^{1/3} \frac{1}{2} \rho(\tau) \cos(t - \alpha t^2 - \Psi(\tau)) + O(\varepsilon^{2/3}); \quad \tau = \varepsilon^{2/3} t. \]
Analytical results for pendulum

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\]

Result of averaging

\[
\frac{d\rho}{d\tau} = f(\tau) \sin \Psi - \gamma \rho, \quad \rho \left[ \frac{d\Psi}{d\tau} + \lambda \tau - \rho^2 \right] = f(\tau) \cos \Psi.
\]
Asymptotic approximation on the initial stage

Original equation

\[
\frac{d^2 x}{dt^2} + \sin x = -\Gamma \frac{dx}{dt} + \varepsilon (1 + \delta t) \cos (t - \alpha t^2), \quad 0 < \varepsilon \ll 1.
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- $$f(\tau) = 1 + (\delta / \varepsilon^{2/3}) \tau, \quad \tau = \varepsilon^{2/3} t$$
Asymptotic approximation on the initial stage

Original equation

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\frac{d^2 x}{dt^2} + \sin x = -\Gamma \frac{dx}{dt} + \varepsilon(1 + \delta t) \cos(t - \alpha t^2), \quad 0 < \varepsilon \ll 1.
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- \( f(\tau) = 1 + (\delta/\varepsilon^{2/3}) \tau, \quad \tau = \varepsilon^{2/3} t \)
- \( \lambda = 2\alpha/\varepsilon^{4/3} \)
Asymptotic approximation on the initial stage

Original equation

\[ \frac{d^2 x}{dt^2} + \sin x = -\Gamma \frac{dx}{dt} + \varepsilon (1 + \delta t) \cos (t - \alpha t^2), \quad 0 < \varepsilon \ll 1. \]

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- \( \lambda = 2\alpha/\varepsilon^{4/3} \)
- \( \gamma = \Gamma/2\varepsilon. \)
Asymptotic approximation on the initial stage

Original equation

\[
\frac{d^2 x}{dt^2} + x - \frac{x^3}{6} = -\Gamma \frac{dx}{dt} + \varepsilon (1 + \delta t) \cos(t - \alpha t^2).
\]

Result of averaging

\[
\frac{d\rho}{d\tau} = f(\tau) \sin \Psi - \gamma \rho, \quad \rho \left[ \frac{d\Psi}{d\tau} + \lambda \tau - \rho^2 \right] = f(\tau) \cos \Psi.
\]

- \( f(\tau) = 1 + \left( \delta / \varepsilon^{2/3} \right) \tau \)
- \( \lambda = 2\alpha / \varepsilon^{4/3} \)
- \( \gamma = \Gamma / 2\varepsilon. \)
Asymptotic approximation on the initial stage

Anzatz on the initial stage

\[ x(t; \varepsilon) = \varepsilon^{1/3} \frac{1}{2} \rho(\tau) \cos(t - \alpha t^2 - \Psi(\tau)) + O(\varepsilon^{2/3}); \quad \tau = \varepsilon^{2/3} t. \]

Reduced equations

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Problem on autoresonance
Asymptotic approximation on the initial stage

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Problem on autoresonance

- Is there any solution with increasing amplitude: \( \rho(\tau) \rightarrow \infty \) as \( \tau \rightarrow \infty \)?
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Problem on autoresonance

- Is there any solution with increasing amplitude: \( \rho(\tau) \to \infty \) as \( \tau \to \infty \)?
- How many solutions of that type exist?
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Problem on autoresonance

- Is there any solution with increasing amplitude: \( \rho(\tau) \to \infty \) as \( \tau \to \infty \)?
- How many solutions of that type exist?
- Are they stable?
Analytical results. Dissipationless system.

\[ \frac{d\rho}{d\tau} = f \sin \psi, \quad \rho \left[ \frac{d\psi}{d\tau} + \lambda \tau - \rho^2 \right] = f \cos \psi, \quad \lambda = \text{const} > 0. \]
Analytical results. Dissipationless system.

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\frac{d\rho}{d\tau} = f \sin \psi, \quad \rho \left[ \frac{d\psi}{d\tau} + \lambda \tau - \rho^2 \right] = f \cos \psi, \quad \lambda = \text{const} > 0.
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**Theorem**

- **Autoresonance solutions.**
  Let be \( \gamma = 0, \ f = \text{const} \). If \( \lambda > 0 \) then there exist two-parametric solution, which has the increasing amplitude \( \rho(\tau) = \sqrt{\lambda \tau} + O(\tau^{-3/8}), \ \tau \to \infty \).
Analytical results. Dissipationless system.

\[ \frac{d \rho}{d \tau} = f \sin \psi, \quad \rho \left[ \frac{d \psi}{d \tau} + \lambda \tau - \rho^2 \right] = f \cos \psi, \quad \lambda = \text{const} > 0. \]

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- **Nonautoresonance solutions.**
  Let be \( \gamma = 0, \ f = \text{const.} \) There always exist two-parametric solution, which has the bounded amplitude \( \rho(\tau) = \mathcal{O}(1). \) If \( \lambda < 0 \) then the amplitude of any solution is bounded.
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Separation of the autoresonance solutions

\[ \frac{d\rho}{d\tau} = f \sin \psi, \quad \rho \left[ \frac{d\psi}{d\tau} + \lambda \tau - \rho^2 \right] = f \cos \psi, \quad f, \lambda = \text{const} > 0. \]

Main problem in autoresonance theory
Separation of the autoresonance solutions

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Separation of the autoresonance solutions

The phase plain at the initial moment (R.Garifullin, 2003)
Separation of the autoresonance solutions

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Dissipation systems.

\[ d^2x/dt^2 + \sin x = -\Gamma dx/dt + \varepsilon \cos (t - \alpha t^2), \quad \Gamma \approx \varepsilon > 0. \]

Is autoresonance possible in dissipation systems?
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Dissipation system under a constant pumping amplitude

Main resonance equations

\[
\frac{d\rho}{d\tau} = f(\tau) \sin \psi - \gamma \rho, \quad \rho \left[ \frac{d\psi}{d\tau} + \lambda \tau - \rho^2 \right] = g(\tau) \cos \psi, \quad \gamma, \lambda > 0.
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Let be \( f = \text{const.} \) If \( \gamma > 0 \) then the amplitude of each solution is bounded.
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**Theorem.**

Let be \( f = \text{const}. \) If \( \gamma > 0 \) then the amplitude of each solution is bounded.

**Corollary.**

Autoresonance phenomenon can not be in any dissipation system under constant driver amplitude.
Constant pumping amplitude. Damping under dissipation

**Strong dissipation** $\gamma = 1$

![Graph showing damping over time](image)
Constant pumping amplitude. Damping under dissipation

Weak dissipation $0 < \gamma \ll 1$
In order to compensate dissipation losses of energy and to keep system in the autoresonant mode we offer to increase slowly the pumping amplitude:

\[
\begin{align*}
\frac{d^2 x}{dt^2} + \sin x &= -\Gamma \frac{dx}{dt} + \varepsilon (1 + \delta t) \cos (t - \alpha t^2), \\
\delta &\approx \varepsilon^2/3.
\end{align*}
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Resonance under dissipation $\Gamma > 0$. Idea.

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Under \( f_1 = \delta / \varepsilon^{2/3} \), \( \lambda = 2\alpha / \varepsilon^{4/3} \), \( \gamma = \Gamma / 2\varepsilon \) and slow time \( \tau = \varepsilon^{2/3} t \):

**Averaged equations**

\[
\frac{d\rho}{d\tau} = -(1 + f_1 \tau) \sin \psi - \gamma \rho, \quad \rho \left[ \frac{d\psi}{d\tau} + \lambda \tau - \rho^2 \right] = (1 + f_1 \tau) \cos \psi.
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Dissipation system under the varying pumping amplitude

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Theorem

- Autoresonance solutions. Let be \( \gamma > 0 \) and \( f_1 \neq 0 \). If \( \lambda > 0 \) then there exists two-parametric solution, which has the increasing amplitude \( \rho(\tau) = \sqrt{\lambda \tau} [1 + o(1)], \ \tau \to \infty. \)
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  Let be \( \gamma > 0 \). There always exists two-parametric solution, which has the bounded amplitude \( \rho(\tau) = O(1) \).
Autoresonance and dissipation

Increase of the driver amplitude: \( f_1 = 0.25 \). Nonautoresonance.
Autoresonance and dissipation

Increase of the driver amplitude: $f_1 = 0.3$. Autoresonance.
Autoresonance and dissipation

Increase of the driver amplitude: $f_1 = 0.35$. Autoresonance.
Conclusions for the pendulum

We proved the existence of the autoresonance phenomenon in the oscillating systems which have a small dissipation. Increase of the small driver amplitude is necessary for existence of this phenomenon. Increase of the output amplitude is determined by variation of the driver frequency. Ensemble of the autoresonance solutions depends on the rate of growth of the driver amplitude.
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Difficulty of the asymptotics

\[
\begin{align*}
\rho(t) &= \lambda t + \sum_{k=0}^{\infty} \rho_k t^{k/2}, \\
\Psi(t) &= \Psi_0 + \sum_{k=1}^{\infty} \Psi_k t^{k/2},
\end{align*}
\]

Two different solutions

\[
\sin \Psi_0 = 0 \Rightarrow \text{either } \Psi_0 = 0 \text{ or } \Psi_0 = \pi.
\]

There are no any integration constants in the expansions.
Dificulty of the asymptotics

Back to the main resonance equations

\[
\frac{d\rho}{dt} = \sin \psi, \quad \rho \left[ \frac{d\psi}{dt} - \rho^2 + \lambda t \right] = \cos \psi.
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Power asymptotics

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Asymptotics is a beautiful and very hard task

Two parametric WKB-type asymptotics

\[ \rho(t; c, s_0) = \sqrt{\lambda} t + t^{-1/4} \sum_{n=1}^{\infty} t^{-n/8} \rho_n(S; c), \]

\[ \psi(t; c, s_0) = \pi + \sum_{n=1}^{\infty} t^{-n/8} \psi_n(S; c), \quad t \to \infty. \]

\[ S = (4/5) \sqrt{2\lambda}^{1/4} t^{5/4} + s_{0,1}(c) \ln t + s_0 \]
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Construction of the WKB-asymptotics is very hard. It is the art in some sense (V. Babich)
Difficulty of the asymptotics

Equations of degenerate resonance

\[ \frac{d\rho}{dt} = \sin \psi, \quad \frac{d\psi}{dt} = \rho^2 - \lambda t. \]

There are no any WKB-asymptotics at all.
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Dissipation effect

\[
\frac{d \rho}{dt} = (f_0 + f_1 t) \sin \psi - \gamma \rho, \quad \frac{d \psi}{dt} = \rho^2 - \lambda t.
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WKB-asymptotics ???
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Parametric autoresonance equations

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\frac{d\rho}{dt} = \rho \sin \Psi, \quad \frac{d\psi}{dt} - \rho + \lambda t = b \cos \Psi.
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First correction WKB-asymptotics is known.
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WKB-asymptotics ???
Stability of the autoresonance

Physicist:
There is only interest in the stable solutions for applications. Those solutions correspond to physically realizable states.
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Skill of Lyapunov functions instead of skill of asymptotics

Idea is to prove stability of a single power solution instead the general WKB-asymptotics.
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Skill of Lyapunov functions instead of skill of asymptotics

Idea is to prove stability of a single power solution instead the general WKB-asymptotics.

This approach is more preferable for physicists, because it allows to take into account the random perturbation.
Example: Bloch equations

\[\begin{align*}
\frac{d\rho}{dt} &= -t \sin \Psi - \beta_2 \rho, \\
\frac{dz}{dt} &= t \rho \sin \Psi - \beta_1 z \\
\rho \left[ \frac{d\psi}{dt} + \lambda t - z \right] &= -t \cos \Psi; \quad (\lambda, \beta_1, \beta_2 = \text{const} > 0).
\end{align*}\]
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Bad idea is to construct three parametric WKB-asymptotics with growing amplitude.

Alternative approach is to prove the stability of a single solution with growing amplitude.
Averaged Bloch equations

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\begin{align*}
\frac{d\rho}{dt} &= -t \sin \psi - \beta_2 \rho, \\
\frac{dz}{dt} &= t \rho \sin \psi - \beta_1 z \\
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\end{align*}
\]

Power asymptotics

\[
\begin{align*}
\rho(t) &= r_1 \sqrt{t} + r_0 + \sum_{n=1}^{\infty} \rho_n t^{-n/2}, \\
z(t) &= \lambda t + z_1 \sqrt{t} + z_0 + \sum_{n=1}^{\infty} z_n t^{-n/2}, \\
\psi(t) &= \psi_0 + \sum_{n=1}^{\infty} \psi_n t^{-n/2}.
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Power asymptotics

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Leading order term

\[ \sin \psi_0 = 0 \Rightarrow \psi_0 = \pi, \text{ or } \psi_0 = 0 \Rightarrow \text{ There are two asymptotic solutions with growing amplitude.} \]
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What solution is stable?
**Main result**

\[
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Denote the solution \(R_0(t), Z_0(t), \psi_0(t)\) which is determined by the leading order term \(\psi_0 = 0\).
Main result

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Denote the solution \( R_0(t), Z_0(t), \psi_0(t) \) which is determined by the leading order term \( \psi_0 = 0 \).

**Theorem**

If \( \beta_1, \beta_2 > 0 \) then the solution \( R_0(t), Z_0(t), \psi_0(t) \) is asymptotically stable as \( t \to \infty \).
Main result

Construction of a Lyapunov’s function for the remainder problem is the idea of the proof.
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Construction of a Lyapunov’s function for the remainder problem is the idea of the proof.

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Future: Stability of the power solutions for different autoresonance models.
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References

Future activity
Stability of autoresonance models under random perturbation.
Initial stage of autoresonance. Zero dissipation $\beta = 0$.

Constant driver amplitude: $f_1 = 0$. Autoresonance solution.
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