# On birational nature of Isomonodromic Deformation equations

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**Аннотация**

I propose the transparent geometric model that makes possible to present the set of the rational Darboux coordinates on the phase space of the Isomonodromic Deformation equation.

In the foundation of the construction lie an observation that for the matrix from the orbit the projections of the kernel and image to the corresponding complementary coordinate subspaces are conjugated each other with respect to the canonical structure of the orbit.

It is known that the famous Painlevé VI equation has surprisingly rich group of birational symmetries. The equation describes isomonodromic deformations of $2 \times 2$ Fuchsian system with four poles.

The generic case of $N \times N$ matrices with different eigenvalues has the similar birational symmetries. What about a general case? Are there the same birational symmetries in the degenerated case of low-dimensional orbits with multiple eigenvalues, or the situation is similar to the difference between the twisted and the plane cubics, where the first one is rational and the second one is not?

I will show that the phase spaces of the Isomonodromic Deformation equations have the same structure of the birational symplectic manifold in the degenerated cases too, at least if there is enough number of one-dimensional eigenspaces. The possibility to define the rational canonical variables on the same system in several ways, like the permutations of the basic vectors or renumbering the poles, is the source of the birational symmetries in question.

Let us consider the deformation of the Fuchs equation

$$\frac{d}{dz}\Psi = \sum_{k=0}^{M} \frac{A^{(k)}}{z - z^k}\Psi;\ \ A^{(k)} \in \mathrm{sl}(N,\mathbb{C});\ \ z, z^k \in \mathbb{C}. \tag{1}$$

It is known that the isomonodromic deformation of this equation may be associated with some Hamiltonian system defined on the space that we denote by $\mathcal{O}_{J^1}\times\mathcal{O}_{J^2}\times\cdots\times\mathcal{O}_{J^M}/\!\!/\,\mathrm{PGL}(N,\mathbb{C})$. This space is the quotient of the product of several (co)adjoint orbits $\mathcal{O}_{J^k} := \cup_{g\in PGL(N,\mathbb{C})} gJ^{(k)}g^{-1} \ni A^{(k)}$ over the diagonal (co)adjoint action of $\mathrm{GL}(N,\mathbb{C})$ intersected by the momentum level $\Sigma := \sum_{k=1}^{M} A^{(k)} = 0$.

Let us built a set of the canonical coordinates on an orbit first. The construction is based on the possibility to project a linear transformation $A \in \operatorname{End} V$ along its eigenspace $\ker(A - \lambda_1 I) \neq 0$ to $\operatorname{End} V/\ker(A - \lambda_1 I)$. The Jordan structure of the projection is defined by the Jordan structure of $A$, all the Jordan chains corresponding to $\lambda_1$ become one unit shorter. The fiber of the projection is the linear symplectic space, so after the introducing a basis in $V$ we get the symplectic fiberation of the orbit. The iteration of the construction gives the birational symplectomorphism between the orbit $\mathcal{O}_J$ and the linear symplectic space with the natural Darboux coordinates.

To parameterize the Isomonodromic Deformation phase space $\mathcal{O}_{J^{(0)}} \times \mathcal{O}_{J^{(1)}} \times \cdots \times \mathcal{O}_{J^{(M)}}/\!\!/\,\mathrm{PGL}(N,\mathbb{C})$ it is possible to construct a basis $\mathbf{e} := e^1,\ldots,e^N$ rigidly connected with the set of $A^{(0)},\ldots A^{(M)} := \bar{A}$:

$$\mathbf{e}(g^{-1}\bar{A}g) = \mathbf{e}(\bar{A})g.$$

It is equivalent to the factorization with respect to the diagonal adjoint action of $\mathrm{GL}(N,\mathbb{C})$. The problem is to control the momentum map $\Sigma = \sum_{k=0}^{M} A^{(k)}$.

I will present the iteration procedure for the construction of the basis $\mathbf{e}$ with the necessary properties. The construction is based on the following observation.

Let we project each of the transformations $A^{(k)} \in \operatorname{End} V$ along *its own* fixed in someway one-dimensional subspace $K_1^{(k)}$ of the eigenspace $\ker(A^{(k)} - \lambda_1^{(k)}) \supset K_1^{(k)}$ on one hyper-subspace $V_1 \subset V$, $\dim V_1 = \dim V - 1$. Denote such a projections by $A_1^{(k)}$. Consider the difference $\sigma_1$ between two transformations of $V_1$. The first one is the projection back to $V_1$ along any fixed direction

$e_1^1$ of the constriction $\Sigma|_{V_1}$. The second one is the sum of the projections $\sum_{k=1}^M A_1^{(k)} = \Sigma_1$. The observation is: *the transformation* $\sigma_1 \in \mathrm{End}\, V_1$ *depends on the directions* $\ker(A^{(k)} - \lambda_1^{(1)} I)$, $\mathrm{im}(A^{(k)} - \lambda_1^{(1)} I)$ *and* $e_1^1$ *only.*

# 0 Examples

## 0.1 The $2 \times 2$ case that is the Painlevé VI-case.

Let us denote

$$\mathcal{A}\left(\begin{array}{c} p \\ q \end{array} | -\lambda\lambda\right) := \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \begin{pmatrix} -\lambda & q \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix} = \begin{pmatrix} -(\lambda+pq) & q \\ -p(pq+2\lambda) & pq+\lambda \end{pmatrix}.$$

We put $A^{(1)} = \mathcal{A}\left(\begin{array}{c} p \\ q \end{array} | -\lambda\lambda\right)$, $A^{(0)} = \mathcal{A}\left(\begin{array}{c} z \\ 1 \end{array} | -\mu\mu\right)$, and

$$A^{(2)} = \begin{pmatrix} -\eta & y \\ 0 & \eta \end{pmatrix}, \quad A^{(3)} = \begin{pmatrix} -\nu & 0 \\ x & \nu \end{pmatrix}.$$

It is evident that the symplectic form $\omega = \omega^{(1)} + \omega^{(2)} + \omega^{(3)} + \omega^{(0)}$ on the quotient space is equal to $dp \wedge dq$, because $\omega^{(2)} = \omega^{(3)} = \omega^{(0)} = 0$. The momentum level-set equation $\sum_k A^{(k)} = 0$ can be trivially solved because $x, y$ and $z$ are summands of the anti-diagonal and diagonal matrix elements of the momentum $\sum_k A^{(k)}$. We get:

$$A^{(1)} = \begin{pmatrix} -(\lambda+pq) & q \\ -p(pq+2\lambda) & pq+\lambda \end{pmatrix}, A^{(0)} = \begin{pmatrix} pq+\sigma & 1 \\ -(pq+\sigma+\mu)(pq+\sigma-\mu) & -(pq+\sigma) \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} -\eta & -(q+1) \\ 0 & \eta \end{pmatrix}, A^{(3)} = \begin{pmatrix} -\nu & 0 \\ p^2q(q+1) + 2p(q\sigma+\lambda) + \sigma^2 - \mu^2 & \nu \end{pmatrix},$$

where by $\sigma$ we denote the sum of all parameters (eigenvalues): $\sigma = \lambda + \mu + \nu + \eta$.

We can see that $\mathrm{tr}(A^{(1)} + tA^{(0)})A^{(3)} =$

$$= p^2q(q+1)(q+t) + 2pq(q+1)(q+t)\left(\frac{\lambda}{q} + \frac{\sigma-\nu-\lambda}{q+1} + \frac{\nu}{q+t}\right) + (q+t)(\sigma^2-\mu^2)$$

It is the Hamiltonian $H(-p,-q,t)t(t-1) = \mathrm{tr}(A^{(1)} + tA^{(0)})A^{(3)}$ of the Painlevé VI equation with the parameters

$$\alpha = 2((2\lambda-\sigma)^2 + \sigma^2 - \mu^2), \beta = 2(\mu+\eta)^2, \gamma = 2\lambda^2, \delta = 2(\nu-1/2)^2.$$

Painlevé VI equation is satisfied by the function $-q(t)$.

## 0.2 The generic $3 \times 3$ case.

If we follow the notations of the Painlevé VI case strictly, we get the Hamiltonian qubical in variables $p_i$'s. To my taste the following choice of the notations is a little bit more observable and symmetric...

Let us denote $\mathcal{A}\left(\begin{array}{ccc} q_1 & q_2 & q_3 \\ p_1 & p_2 & p_3 \end{array} |\lambda_1\lambda_2\lambda_3\right) =$

$$= \begin{pmatrix} 1 & 0 & 0 \\ q_1 & 1 & 0 \\ q_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & p_1 \, p_2 \\ 0 & \\ 0 & \mathcal{A}_{2\times 2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -q_1 & 1 & 0 \\ -q_2 & 0 & 1 \end{pmatrix},$$

where

$$\mathcal{A}_{2\times 2} = \begin{pmatrix} 1 & 0 \\ q_3 & 1 \end{pmatrix} \begin{pmatrix} \lambda_2 & p_3 \\ 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -q_3 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_2 - p_3 q_3 & p_3 \\ -(p_3 q_3 - \lambda_2 + \lambda_3) q_3 & \lambda_3 + p_3 q_3 \end{pmatrix}$$

Let us split $\mathcal{A}$ on the upper-, lower-, and diagonal parts, and introduce three 3-dimensional vector-columns:

$$A_{\Uparrow} := \begin{pmatrix} p_1 \\ p_2 \\ p_3 + p_2 q_1 \end{pmatrix} =: \begin{pmatrix} (A_{\Uparrow})_1 \\ (A_{\Uparrow})_2 \\ (A_{\Uparrow})_3 \end{pmatrix}, A_{\Delta} := \begin{pmatrix} -q_1 p_1 - q_2 p_2 \\ q_1 p_1 - q_3 p_3 \\ q_2 p_2 + q_3 p_3 \end{pmatrix} =: \begin{pmatrix} (A_{\Delta})_1 \\ (A_{\Delta})_2 \\ (A_{\Delta})_3 \end{pmatrix}$$

$$A_{\Downarrow}^{\lambda} := \begin{pmatrix} -p_1 q_1^2 - p_2 q_1 q_2 + p_3(q_1 q_3 - q_2) + q_1(\lambda_1 - \lambda_2) \\ -p_1 q_1 q_2 - p_2 q_2^2 + p_3 q_3 (q_1 q_3 - q_2) + q_1 q_3 (\lambda_3 - \lambda_2) + q_2(\lambda_1 - \lambda_3) \\ p_1 q_2 - p_3 q_3^2 + q_3(\lambda_2 - \lambda_3) \end{pmatrix}$$

$$=: \begin{pmatrix} (A_{\Downarrow})_1 \\ (A_{\Downarrow})_2 \\ (A_{\Downarrow})_3 \end{pmatrix}$$

$$\mathcal{A}\left(\begin{array}{ccc} q_1 & q_2 & q_3 \\ p_1 & p_2 & p_3 \end{array} |\lambda_1\lambda_2\lambda_3\right) = \begin{pmatrix} (A_{\Delta})_1 + \lambda_1 & (A_{\Uparrow})_1 & (A_{\Uparrow})_2 \\ (A_{\Downarrow}^{\lambda})_1 & (A_{\Delta})_2 + \lambda_2 & (A_{\Uparrow})_3 \\ (A_{\Downarrow}^{\lambda})_2 & (A_{\Downarrow}^{\lambda})_3 & (A_{\Delta})_3 + \lambda_3 \end{pmatrix}$$

The standard pairing $\operatorname{tr} AB$ has the representation

$$\operatorname{tr} AB =< A_{\Downarrow}^{\lambda}, B_{\Uparrow} > + < A_{\Uparrow}, B_{\Downarrow}^{\mu} > + < A_{\Delta} + \vec{\lambda}, B_{\Delta} + \vec{\mu} >,$$

where matrix $B$ from the orbit of $\mathrm{diag}(\mu_1\mu_2\mu_3)$ is splited in the same way as $A$ is, and $<,>$ is the standard Euclidean scalar product on $\mathbb{C}^3$.

Consider four matrices $A^{(1)}, A^{(2)}, A^{(3)}$ and $A^{(0)}$ from the orbits of $J^{(1)} = \mathrm{diag}(\lambda_1\lambda_2\lambda_3)$, $J^{(2)} = \mathrm{diag}(\eta_1\eta_2\eta_3)$, $J^{(3)} = \mathrm{diag}(\nu_1\nu_2\nu_3)$ and $J^{(0)} = \mathrm{diag}(\mu_1\mu_2\mu_3)$ correspondingly. We assume that $\sum_i \lambda_i = \sum_i \eta_i = \sum_i \nu_i = \sum_i \mu_i = 0$.

In the constructed on the lecture frame of reference, $A^{(2)}$ is upper-triangular, $A^{(3)}$ is lower-triangular and $A^{(0)}$ has the fixed eigenvector $(111)^T$ corresponding to $\mu_1$.

The symplectic form on the quotient space is $\omega = \sum_{i=0}^{3} dp_i \wedge dq_i$, where functions $p_1, q_1, p_2, q_2, p_3, q_3$ parameterize the orbit $\mathcal{O}_{J^{(1)}}$ and $p_0, q_0$ parameterize the projection of $\mathcal{O}_{J^{(0)}}$ on the two-dimensional orbit of $\mathrm{diag}(\mu_2, \mu_3) \in \mathrm{gl}(2)$.

We denote $A^{(1)}$ and $A^{(0)}$ by $A + \mathrm{diag}(\lambda_1\lambda_2\lambda_3)$ and $B + \mathrm{diag}(\mu_1\mu_2\mu_3)$:

$$A^{(1)} = \mathcal{A}\left( \begin{array}{ccc} q_1 & q_2 & q_3 \\ p_1 & p_2 & p_3 \end{array} |\lambda_1\lambda_2\lambda_3 \right) =: A + \mathrm{diag}(\lambda_1\lambda_2\lambda_3),$$

$$A^{(0)} = \mathcal{A}\left( \begin{array}{ccc} 1 & 1 & q_0 \\ z_1 & z_2 & p_0 \end{array} |\mu_1\mu_2\mu_3 \right) =: B + \mathrm{diag}(\mu_1\mu_2\mu_3).$$

Several more notations:

$$A^{(2)} = \begin{pmatrix} \eta_1 & y_1 & y_2 \\ 0 & \eta_2 & y_3 \\ 0 & 0 & \eta_3 \end{pmatrix}, A^{(3)} = \begin{pmatrix} \nu_1 & 0 & 0 \\ x_1 & \nu_2 & 0 \\ x_2 & x_3 & \nu_3 \end{pmatrix}, \ \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\vec{\sigma} = \vec{\lambda} + \vec{\eta} + \vec{\nu} + \vec{\mu} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}, \quad \sigma_1 = -(\sigma_2 + \sigma_3),$$

where vectors $\vec{\lambda}, \vec{\eta}, \vec{\nu}$ and $\vec{\mu}$ are collected from the corresponding eigenvalues.

*Let us solve the momentum-level equation* $\sum_k A^{(k)} = 0$ *now.*

Consider the diagonal part of the momentum-level equation first. It will be satisfied if we fix the values of $z_1$ and $z_2$:

$$A_\Delta + B_\Delta + \vec{\sigma} = 0 \Leftrightarrow z_1 = q_0p_0 - q_1p_1 + q_3p_3 - \sigma_2, z_2 = -(q_0p_0 + q_2p_2 + q_3p_3 + \sigma_3).$$

We satisfy upper- and lower- triangular parts of the $3 \times 3$-matrix equation $\sum_k A^{(k)} = 0$ using $\vec{x}$ and $\vec{y}$:

$$-\vec{x} = B^{\mu}_{\Downarrow} + A^{\lambda}_{\Downarrow}, \quad -\vec{y} = A_{\Uparrow} + B_{\Uparrow},$$

where $B$ is calculated using $z_1$ and $z_2$: $B + \mathrm{diag}(\mu_1\mu_2\mu_3) =$

$$= \mathcal{A}\begin{pmatrix} 1 & 1 & q_0 & |\mu_1\mu_2\mu_3 \\ q_0p_0 - q_1p_1 + q_3p_3 - \sigma_2 & -(q_0p_0 + q_2p_2 + q_3p_3 + \sigma_3) & p_0 & \end{pmatrix}.$$

To get the Hamiltonian we have to calculate $\mathrm{tr}(A^{(1)}t + A^{(0)})A^{(3)} := H$. In our notations it is $< A_\Uparrow t + B_\Uparrow, \vec{x} > + < A_\Delta t + \vec{\lambda} t + B_\Delta + \vec{\mu}, \vec{\nu} >$, consequently

$$H = - < A_\Uparrow t + B_\Uparrow, A^\lambda_\Downarrow + B^\mu_\Downarrow > + (t-1) < A_\Delta, \vec{\nu} > + const_t.$$

We use $tA_\Delta + B_\Delta = tA_\Delta - A_\Delta - \vec{\sigma} = (t-1)A_\Delta - \vec{\sigma}$ here.

$$< A_\Delta, \nu >= (2\nu_2 + \nu_3)p_1q_1 + (\nu_2 + 2\nu_3)p_2q_2 + (\nu_3 - \nu_2)p_3q_3 + const$$

$$A_\Uparrow t + B_\Uparrow = \begin{pmatrix} p_0q_0 - p_1(q_1 - t) + p_3q_3 - \sigma_2 \\ -p_0q_0 - p_2(q_2 - t) - p_3q_3 - \sigma_3 \\ -p_0(q_0 - 1) - p_2(q_2 - q_1t) - p_3(q_3 - t) - \sigma_3 \end{pmatrix}.$$

To finish the calculations we have to multiply this $A_\Uparrow t + B_\Uparrow$ on $A^\lambda_\Downarrow + B^\mu_\Downarrow$:

$$(A^\lambda_\Downarrow + B^\mu_\Downarrow)_1 = p_0(q_0 - 1) + q_1p_1(1 - q_1) + q_2p_2(1 - q_1) + p_3(q_1q_3 - q_2) + \\ + q_1(\lambda_1 - \lambda_2) + \sigma_2 + \sigma_3 + \mu_1 - \mu_2$$

$$(A^\lambda_\Downarrow + B^\mu_\Downarrow)_2 = q_0q_0(q_0 - 1) + p_1q_1(1 - q_2) + p_2q_2(1 - q_2) + p_3q_3(q_1q_3 - q_2) + \\ + q_1q_3(\lambda_3 - \lambda_2) + q_0(\mu_3 - \mu_2) + q_2(\lambda_1 - \lambda_3) + \sigma_2 + \sigma_3 + \mu_1 - \mu_3$$

$$(A^\lambda_\Downarrow + B^\mu_\Downarrow)_3 = -p_0q_0(q_0 - 1) + p_1(-q_1 + q_2) + p_3q_3(-q_3 + 1) + \\ + q_0(\mu_2 - \mu_3) + q_3(\lambda_2 - \lambda_3) - \sigma_2$$

So, *in the generic $3 \times 3$ case the Isomonodromic Deformation equations are the Euler-Lagrange equations for*

$$dp_1 \wedge dq_1 + dp_2 \wedge dq_2 + dp_3 \wedge dq_3 + dp_0 \wedge dq_0 - dH \wedge \frac{dt}{t(1-t)},$$

*where* $H =$

$$
\begin{aligned}
&= (-p_0q_0 + p_1(q_1 - t) - p_3q_3 + \sigma_2)(p_0(q_0 - 1) + p_1q_1(1 - q_1) + p_2q_2(1 - q_1) + \\
&\quad + p_3(q_1q_3 - q_2) + q_1(\lambda_1 - \lambda_2) + \sigma_2 + \sigma_3 + \mu_1 - \mu_2) + \\
&\quad + (p_0q_0 + p_2(q_2 - t) + p_3q_3 - \sigma_3)(p_0q_0(q_0 - 1) + p_1q_1(1 - q_2) + p_2q_2(1 - q_2) + \\
&\quad + p_3q_3(q_1q_3 - q_2) + q_1q_3(\lambda_3 - \lambda_2) + q_0(\mu_3 - \mu_2) + q_2(\lambda_1 - \lambda_3) + \sigma_2 + \sigma_3 + \mu_1 - \mu_3) + \\
&\quad + (p_0(q_0 - 1) + p_2(q_2 - q_1t) + p_3(q_3 - t) + \sigma_3)(p_0q_0(q_0 - 1) - p_1(-q_1 + q_2) + \\
&\quad + p_3q_3(q_3 - 1) + + q_0(\mu_2 - \mu_3) + q_3(\lambda_2 - \lambda_3) - \sigma_2) + \\
&\quad + (t - 1)((2\nu_2 + \nu_3)q_1p_1 + (\nu_2 + 2\nu_3)q_2p_2 + (\nu_3 - \nu_2)q_3p_3).
\end{aligned}
$$

We can see that polynomial $H(p, q, t)$ is linear in $t$ and quadratic with respect to $p$-variables:

$$\sum_{i,j=0}^{3} \mathcal{P}_{ij}^{II}(q)p_ip_j + 2\sum_{k=0}^{3} \mathcal{P}_k^I(q, \lambda\mu\nu\eta)p_k + \mathcal{P}^0(q, \lambda\mu\nu\eta).$$

All $\mathcal{P}$-s are polynomials linear in $t$. The constant term $\mathcal{P}^0$ is quadratic with respect to parameters $\lambda_i, \mu_i, \nu_i, \eta_i$, $\mathcal{P}^I$ is linear with respect to the parameters and the quadratic therm $\mathcal{P}_{ij}^{II}p_ip_j$ does not depend on the parameters.