Maciej Dunajski

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and
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University of Cambridge
Mercator’s navigation

- **Mercator 1569.** Rhumb lines - not the shortest, but easy to follow.
Mercator’s navigation

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- Mercator paths are unparametrised geodesics = images of great circles on a sphere.
A problem of Roger Liouville (1889)

Cover a plane with curves, one curve through each point in each direction. How can you tell whether these curves are geodesics of some metric?
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- Path geometry: $y'' = F(x, y, y')$. (Douglas 1936).

Unparametrised geodesics of affine connection:

$\frac{\partial}{\partial y'} F^{\alpha} = 0$

Levi–Civita connection of $g = E dx^2 + 2 F dx dy + G dy^2$?

Bryant, MD, Eastwood (J. Diff. Geom 2009). Solution to the Liouville's problem: Necessary and sufficient conditions for a family of paths to extremize a distance.

Twistor theory

Twistor space

Point

O(1) curve

Geodesic

U
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- Twistor theory
Parallel at latitude $\theta$ is stretched by $\cos(\theta)^{-1}$

\[ y = \varphi, \quad x = \int \cos(\theta) - 1 \, d\theta = \ln\left(\tan(\theta) + \cos(\theta) - 1\right). \]

Images of great circles = solutions to the Mercator ODE

\[ y'' = \tanh(x) \left(y' + (y')^3\right). \]
Mercator Projection

Parallel at latitude $\theta$ is stretched by $\cos(\theta)^{-1}$

- Conformal condition (straight lines = loxodromes) $dx/d\theta = \cos(\theta)^{-1}$. 

$\frac{x}{y} = \tan(\theta), x = \int \cos(\theta) - 1 d\theta = \ln((\tan(\theta) + \cos(\theta)) - 1)$.

$\frac{g}{dx^2 + dy^2} = \cosh(x) - 2(\frac{dx}{2})^2$. 

Dunajski (DAMTP, Cambridge) 

Liouville’s Problem 

Moscow, December 2011
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- Edward Wright (1599)

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Great circles $|\mathbf{r}| = 1$, $\mathbf{r} \cdot \mathbf{n} = 0 \leftrightarrow$ points $\mathbf{n} \in S^2$. 
Twistor space

- Great circles $|\mathbf{r}| = 1, \mathbf{r} \cdot \mathbf{n} = 0 \iff$ points $\mathbf{n} \in S^2$.

- Projective duality. $[X, Y, Z] \in \mathbb{RP}^2, [P, Q, R] \in \mathbb{RP}^2^*$. 
  
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Dunajski (DAMTP, Cambridge)  Liouville’s Problem  Moscow, December 2011
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\[ XP + YQ +ZR = 0. \]

\[ xp + yq + 1 = 0, \quad p = P/R, q = Q/R, \quad x = X/Z, y = Y/Z. \]
Nonlinear duality. \( H(x, y, p, q) = 0 \)
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Differentiate, eliminate \((p, q)\)

\[ y'' = \mathcal{F}(x, y, y'). \]
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- Complexify: \( C \subset U \) is geodesic iff \( C \cong \mathbb{CP}^1 \subset T \) is rational with normal bundle \( \mathcal{O}(1) \).
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$$y'' = F(x, y, y').$$

- Complexify: $C \subset U$ is geodesic iff $C \cong \mathbb{CP}^1 \subset T$ is rational with normal bundle $\mathcal{O}(1)$.
- $C$ is a metric geodesic iff there exists a preferred section of $\kappa_T^{-2/3}$ where $\kappa_T$ is the holomorphic canonical bundle of $T$. 
A projective structure on an open set $U \subset \mathbb{R}^n$ is an equivalence class of torsion free connections $[\Gamma]$. Two connections $\Gamma$ and $\hat{\Gamma}$ are equivalent if they share the same unparametrised geodesics.
A projective structure on an open set $U \subset \mathbb{R}^n$ is an equivalence class of torsion free connections $[\Gamma]$. Two connections $\Gamma$ and $\hat{\Gamma}$ are equivalent if they share the same unparametrised geodesics.

The geodesic flows project to the same foliation of $\mathbb{P}(TU)$. The analytic expression for this equivalence class is

$$\hat{\Gamma}^c_{ab} = \Gamma^c_{ab} + \delta_a^c \omega_b + \delta_b^c \omega_a, \quad a, b, c = 1, 2, \ldots, n$$

for some one–form $\omega = \omega_a dx^a$. 
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Two dimensions–link with second order ODEs:

$$\ddot{x}_c + \Gamma^c_{ab} \dot{x}_a \dot{x}_b = 0$$
$$d^2 y \frac{d^2 y}{dx^2} = A_3 \left( \frac{dy}{dx} \right)^3 + A_2 \left( \frac{dy}{dx} \right)^2 + A_1 \frac{dy}{dx} + A_0$$

where $A_0 = -\Gamma^2_{11}$, $A_1 = \Gamma^1_{11} - 2\Gamma^2_{12}$, $A_2 = 2\Gamma^1_{12} - \Gamma^2_{22}$, $A_3 = \Gamma^1_{22}$.
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Two dimensions–link with second order ODEs:

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Two dimensions–link with second order ODEs:

1. $\ddot{x}^c + \Gamma^c_{ab} \dot{x}^a \dot{x}^b = 0$, where $x^a(t) = (x(t), y(t))$.
2. Eliminate the parameter $t$: second order ODE

$$\frac{d^2 y}{dx^2} = A_3 \left( \frac{dy}{dx} \right)^3 + A_2 \left( \frac{dy}{dx} \right)^2 + A_1 \left( \frac{dy}{dx} \right) + A_0, \quad A_i = A_i(x, y)$$

where $A_0 = -\Gamma^2_{11}, A_1 = \Gamma^1_{11} - 2\Gamma^2_{12}, A_2 = 2\Gamma^1_{12} - \Gamma^2_{22}, A_3 = \Gamma^1_{22}$. 

Dunajski (DAMTP, Cambridge)
What are the necessary and sufficient local conditions on a connection $\Gamma_{ab}^c$ for the existence of a one form $\omega_a$ and a symmetric non–degenerate tensor $g_{ab}$ such that the projectively equivalent connection $\Gamma_{ab}^c + \delta_a^c \omega_b + \delta_b^c \omega_a$ is the Levi-Civita connection for $g_{ab}$?
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1. **Necessary condition**: obstruction of order 5 in the components of a connection in a projective class. Point invariant for a second order ODE whose integral curves are the geodesics of $[\Gamma]$ or a weighted scalar projective invariant of the projective class.

2. **Sufficient conditions**: In the generic case vanishing of two invariants of order 6.
Metric $g = E(x, y)dx^2 + 2F(x, y)dxdy + G(x, y)dy^2$ gives

\[
A_0 = (E \partial_y E - 2E \partial_x F + F \partial_x E)(EG - F^2)^{-1}/2,
\]

\[
A_1 = (3F \partial_y E + G \partial_x E - 2F \partial_x F - 2E \partial_x G)(EG - F^2)^{-1}/2,
\]

\[
A_2 = (2F \partial_y F + 2G \partial_y E - 3F \partial_x G - E \partial_y G)(EG - F^2)^{-1}/2,
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\[
A_3 = (2G \partial_y F - G \partial_x G - F \partial_y G)(EG - F^2)^{-1}/2, \quad (*)
\]
Prolongation I

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- First order homogeneous differential operator with 1D fibres \( \sigma^0 : J^1(S^2(T^*U)) \longrightarrow J^0(\text{Pr}(U)) \). Differentiating (\( * \)) prolongs this operator to bundle maps \( \sigma^k : J^{k+1}(S^2(T^*U)) \longrightarrow J^k(\text{Pr}(U)) \).
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**Theorem:** Solutions to \((*)\) \(\Leftrightarrow\) parallel section of a linear connection \(D\) on a rank six vector bundle \(\mathcal{E} \rightarrow U\).
### Prolongation II

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- 5-jets. At least a 1D fiber, at most 83D image. First obstruction $M$. 
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6-jets. Dimension $112 - 3 = 109$. The image of the 7-jets of metric structures can have dimension $108 - 1 = 107$. Two more 6th order obstructions $E_1, E_2$. 

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7-jets. The image has codimension 10. Two relations between the first derivatives of $E_1 = E_2 = 0$ and the second derivatives of the 5th order equation $M = 0$. The system is involutive.

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The importance of 6th order conditions

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- The 6th order conditions are satisfied iff

\[ 1728 c^3 - 8856 c^2 - 2100 c + 625 = 0, \]

\[ 7776 c^5 + 19656 c^4 - 21852 c^3 - 42054 c^2 - 28725 c - 11125 = 0. \]
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  \[ 1728c^3 - 8856c^2 - 2100c + 625 = 0, \]
  \[ 7776c^5 + 19656c^4 - 21852c^3 - 42054c^2 - 28725c - 11125 = 0. \]

- These three polynomials do not have a common root. We can make the 5th order obstruction vanish, but the two 6th order obstructions \( E_1, E_2 \) do not vanish.
Large class of systems (Zakharov, . . . , Manakov–Santini–Grinevich.)
Characteristic initial value problem for $dKP$

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\begin{align*}
\text{Dispersionless Lax pair} & = 1\text{-parameter family of projective structures.} \\
L_t & = \partial_y + \lambda \partial_x - u_x \partial_\lambda, \\
M & = \partial_t + \left(\lambda^2 + u\right) \partial_x + \left(u_y - \lambda u_x\right) \partial_\lambda.
\end{align*}

Boundary cond: Asymptotically flat projective structure $u_{xxx} \approx 0$. 

Dunajski (DAMTP, Cambridge)  
Liouville’s Problem  
Moscow, December 2011
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Dunajski (DAMTP, Cambridge) 

Moscow, December 2011
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Thank You.