A Problem of Roger Liouville

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University of Cambridge

Mercator's navigation

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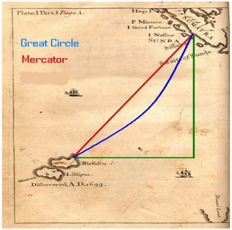

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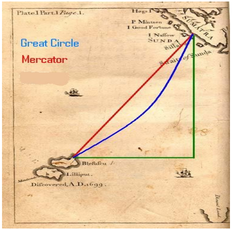
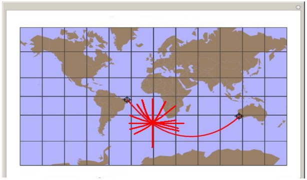


- Mercator paths are unparametrised geodesics = images of great circles on a sphere.



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Cover a plane with curves, one curve through each point in each direction. How can you tell whether these curves are geodesics of some metric?

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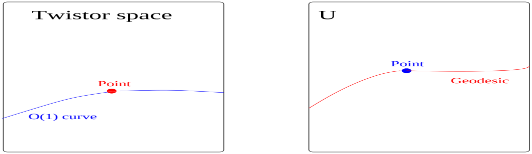
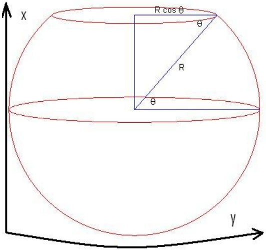
- Twistor theory



Mercator Projection

Parallel at latitude θ is stretched by $\cos(\theta)^{-1}$



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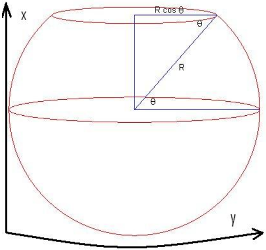

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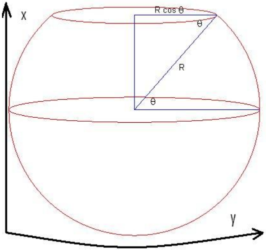

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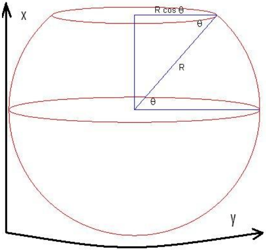

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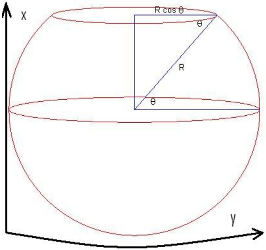

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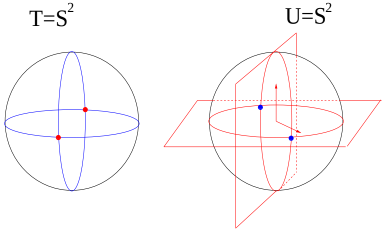
$$y'' = \tanh(x)\Big(y' + (y')^3\Big) \qquad g = \cosh(x)^{-2}(dx^2 + dy^2).$$

Twistor space

- Great circles $|\mathbf{r}| = 1, \mathbf{r} \cdot \mathbf{n} = 0 \longleftrightarrow$ points $\mathbf{n} \in S^2$.



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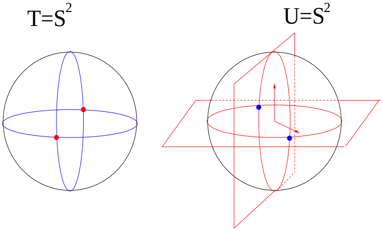

- Projective duality. $[X, Y, Z] \in \mathbb{RP}^2, [P, Q, R] \in \mathbb{RP}^{2^*}$.

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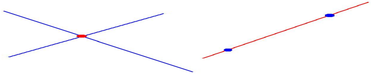

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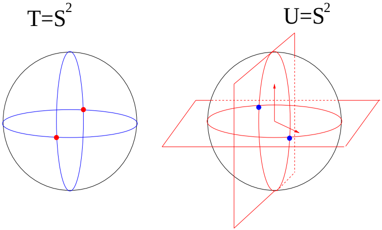

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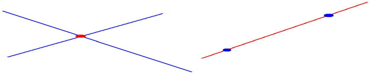

$$xp + yq + 1 = 0, \qquad p = P/R, q = Q/R \quad x = X/Z, y = Y/Z.$$

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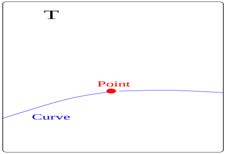
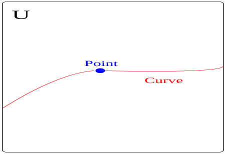
- Nonlinear duality. $H({\color{red}x}, {\color{red}y}, {\color{blue}p}, {\color{blue}q}) = 0$





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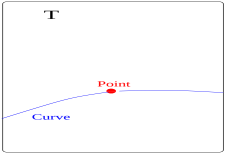

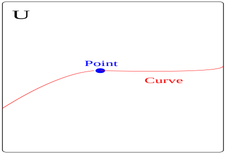

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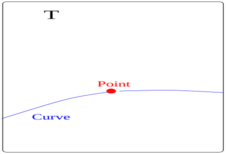

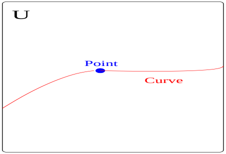

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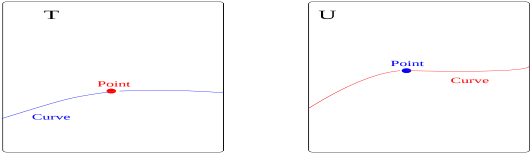

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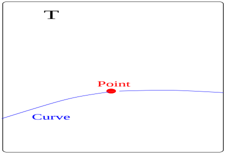

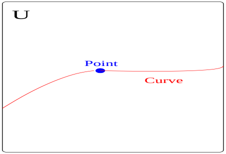

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- C is a metric geodesic iff there exists a preferred section of $\kappa_T{}^{-2/3}$ where κ_T is the holomorphic canonical bundle of T.

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Projective Structures

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- The geodesic flows project to the same foliation of $\mathbb{P}(TU)$. The analytic expression for this equivalence class is

$$\hat{\Gamma}^c_{ab} = \Gamma^c_{ab} + \delta_a{}^c \omega_b + \delta_b{}^c \omega_a, \qquad a, b, c = 1, 2, \dots, n$$

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  2. Eliminate the parameter t: second order ODE

$$\frac{d^2y}{dx^2} = A_3\Big(\frac{dy}{dx}\Big)^3 + A_2\Big(\frac{dy}{dx}\Big)^2 + A_1\Big(\frac{dy}{dx}\Big) + A_0, \quad A_i = A_i(x, y)$$

  where $A_0 = -\Gamma^2_{11}, A_1 = \Gamma^1_{11} - 2\Gamma^2_{12}, A_2 = 2\Gamma^1_{12} - \Gamma^2_{22}, A_3 = \Gamma^1_{22}$.

Metrisability Problem

- What are the necessary and sufficient local conditions on a connection Γ_{ab}^{c} for the existence of a one form ω_a and a symmetric non–degenerate tensor g_{ab} such that the projectively equivalent connection $\Gamma_{ab}^{c} + \delta_a{}^c\omega_b + \delta_b{}^c\omega_a$ is the Levi-Civita connection for g_{ab}?

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2. Sufficient conditions: In the generic case vanishing of two invariants of order 6.

Prolongation I

- Metric $g = E(x,y)dx^2 + 2F(x,y)dxdy + G(x,y)dy^2$ gives

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- First order homogeneous differential operator with 1D fibres $\sigma^0 : J^1(S^2(T^*U)) \longrightarrow J^0(\mathsf{Pr}(U))$. Differentiating $(*)$ prolongs this operator to bundle maps $\sigma^k : J^{k+1}(S^2(T^*U)) \longrightarrow J^k(\mathsf{Pr}(U))$.

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- Theorem: Solutions to $(*)$ $\leftrightarrow$ parallel section of a linear connection D on a rank six vector bundle $\mathcal{E} \to U$.

PROLONGATION II

k	$(J^{k+1}(S^2(T^*U)))$	$(J^k(\mathrm{Pr}(U)))$	$(\ker \sigma^k)$	obstructions
0	9	4	5	0
1	18	12	6	0
2	30	24	6	0
3	45	40	5	0
4	63	60	3	0
5	84	84	1	$1 = \mathbf{1}$
6	108	112	1	$5 = 3 + \mathbf{2}$
7	135	144	1	$10 = 6 + 6 - 2$

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- 7-jets. The image has codimension 10. Two relations between the first derivatives of $E_1 = E_2 = 0$ and the second derivatives of the 5th order equation $M = 0$. The system is involutive.

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The importance of 6th order conditions

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The importance of 6th order conditions

- One parameter family of projective structures

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- These three polynomials do not have a common root. We can make the 5th order obstruction vanish, but the two 6th order obstructions E_1, E_2 do not vanish.

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Thank You.