NONLINEAR SELF-ADJOINTNESS AND CONSERVATION LAWS

Nail H. Ibragimov

Laboratory "Group analysis of mathematical models
in natural and engineering sciences"
Ufa State Aviation Technical University, Ufa, Russia
and Department of Mathematics and Science,
Blekinge Institute of Technology, Karlskrona, Sweden

12 December 2011, Moscow, Russia

Abstract

I introduce the general concept of nonlinear self-adjointness. It embraces author's previous notions of strict self-adjointness and quasi self-adjointness. But the set of nonlinearly self-adjoint equations is essentially wider and includes, in particular, all linear equations. The construction of conservation laws demonstrates a practical significance of the nonlinear self-adjointness. Namely, conservation laws can be associated with symmetries for all nonlinearly self-adjoint systems of differential equations. The system can contain any number of equations. This approach provides a new method for constructing conservation laws and extends Noether's theorem from variational problems to arbitrary systems of differential equations. The new theory is illustrated by various applications.

Definitions and self-adjointness

System of m differential equations

$$F_\alpha\big(x, u, u_{(1)}, \ldots, u_{(s)}\big) = 0, \quad \alpha = 1, \ldots, m. \tag{1}$$

Adjoint equations

$$F^*_\alpha\big(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(s)}, v_{(s)}\big) = 0, \quad \alpha = 1, \ldots, m, \tag{2}$$

with

$$F^*_\alpha\big(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(s)}, v_{(s)}\big) = \frac{\delta \mathcal{L}}{\delta u^\alpha}\,, \tag{3}$$

where $\mathcal{L}$ is the *formal Lagrangian*

$$\mathcal{L} = v^\beta F_\beta \equiv \sum_{\beta=1}^{m} v^\beta F_\beta \tag{4}$$

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \cdots i_s}}\,, \quad \alpha = 1, \ldots, m.$$

Definitions and self-adjointness

Definition

Eqs. (1) are **nonlinearly self-adjoint** if the adjoint equations (2) are satisfied for all solutions of Eqs. (1) upon a substitution

$$v^\alpha = \varphi^\alpha(x, u), \quad \alpha = 1, \ldots, m, \tag{5}$$

with

$$\varphi(x, u) \neq 0.$$

Definitions and self-adjointness

Example

The adjoint equation to the KdV equation

$$u_t = u_{xxx} + uu_x \tag{6}$$

has the form

$$v_t = v_{xxx} + uv_x \tag{7}$$

It is clear that

$$v = u$$

solves the adjoint equation for all solutions of the KdV equation (6). The general substitution of the form (5) is given by

$$v = A_1 + A_2 u + A_3(x + tu) \tag{8}$$

Explicit formula for conserved vectors

Theorem

Let the system of differential equations (1) be nonlinearly self-adjoint. If Eqs. (1) have a Lie point, contact, Lie-Bäcklund or nonlocal symmetry

$$X = \xi^i(x, u, u_{(1)}, \dots)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u, u_{(1)}, \dots)\frac{\partial}{\partial u^\alpha} \qquad (9)$$

then the conserved vector is

$$C^i = W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j\left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha}\right) + D_j D_k\left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\right) - \dots\right] \qquad (10)$$

$$+ D_j\left(W^\alpha\right)\left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k\left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\right) + \dots\right] + D_j D_k\left(W^\alpha\right)\left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \dots\right],$$

Explicit formula for conserved vectors

where

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha \tag{11}$$

and $\mathcal{L}$ is the *formal Lagrangian*.

REMARK 1.

The method is applicable independently on the number of equations in the system and the number of dependent variables.

REMARK 2.

The above Theorem, unlike the Noether theorem, does not require additional restrictions such as the invariance condition or the divergence condition.

Short pulse equation

Example

Ultra-short light pulses in media with nonlinearities

$$D_t D_x(u) = u + \frac{1}{6}\, D_x^2(u^3) \tag{12}$$

Formal Lagrangian

$$\mathcal{L} = v\left[u_{xt} - u - \frac{1}{2}\, u^2 u_{xx} - u u_x^2\right]. \tag{13}$$

Adjoint equation

$$v_{xt} = v + \frac{1}{2}\, u^2 v_{xx}. \tag{14}$$

Eq. (12) is not nonlinearly self-adjoint with a substitution

$$v = \varphi(t, x, u) \tag{15}$$

Short pulse equation

but it is nonlinearly self-adjoint with the differential substitution

$$v = u_t - \frac{1}{2}\, u^2 u_x. \tag{16}$$

Eq. (12) has three symmetries:

$$X_1 = \frac{\partial}{\partial t}\,, \quad X_2 = \frac{\partial}{\partial x}\,, \quad X_3 = u\frac{\partial}{\partial u} + x\frac{\partial}{\partial x} - t\frac{\partial}{\partial t}\,. \tag{17}$$

The symmetries X_1 and X_2 lead to trivial conservation laws, but X_3 gives the nontrivial conservation law

$$D_t\left(u^2\right) + D_x\left(u^2 u_x u_t - u_t^2 - \frac{1}{4}u^4 - \frac{1}{4}u^4 u_x^2\right) = 0. \tag{18}$$