Factorization in Categories of Systems of Linear Partial Differential Equations

Sergey P. Tsarev

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Siberian Federal University, Krasnoyarsk

СИБИРСКИЙ ФЕДЕРАЛЬНЫЙ УНИВЕРСИТЕТ
SIBERIAN FEDERAL UNIVERSITY

Elliptic Moutard transformations

The two-dimensional stationary Schrödinger equation:

$$-(u_{xx}+u_{yy})+V(x,y)u=0 \tag{1}$$

$$-(u_{xx}+u_{yy})+V(x,y)u+Eu=0 \tag{2}$$

Examples:
1) $V(x,y)=0$: harmonic functions $u(x,y)$.
2) $V(x,y)=-k^2=-E$:
$u(x,y)=\sin(ax+by+c)$, $a^2+b^2=k^2$
gives a "basis" of solutions of (1).
Question: How can one construct potentials $V(x,y)$ with some "sufficiently big" set of solutions of (1) or (2) in "explicit form"?

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Known examples:
0) Harmonic oscillator $V = a(x^2 + y^2)$ and a few other textbook examples.
1) Quasi-exactly solvable V (Turbiner A.V., Gonzalez-Lopez A., Kamran N., Olver P. ...) Solutions are constructible for some levels E of the discrete spectrum.
2) D-integrable potentials $V = \sum_{k=1}^{m} \frac{(\alpha_k, \alpha_k)A_k}{((\alpha_k, \vec{x}) + c_k)^2}$
(Berest Yu.Yu.,
Chalykh O., Veselov A. P. ...).
Problem: singularities.
3) Miltidimensional “finite-gap” potentials (Dubrovin B. A., Krichever I.M., Novikov S.P., Veselov A.P., Buchstaber V.M., Enolskii V.Z. ...) V and u are given in terms of the theta-function or the Kleinian hyperelliptic functions.

Elliptic Moutard transformations

Suppose we have 2 solutions $u = \omega$, $u = \phi$ of stationary Schrödinger equation:

$$-(u_{xx} + u_{yy}) + V(x, y)u = 0. \tag{3}$$

Then one can find a solution $u = \theta$ for (3) with another potential

$$V_1 = V - 2\Delta \ln \omega = -V + \frac{2(\omega_x^2 + \omega_y^2)}{\omega^2}, \tag{4}$$

as a solution of the following system:

$$\begin{cases} (\omega\theta)_x = -\omega^2 \left(\frac{\phi}{\omega}\right)_y, \\ (\omega\theta)_y = \omega^2 \left(\frac{\phi}{\omega}\right)_x, \end{cases} \tag{5}$$

Important examples (I.Taimanov, S.Ts., 2009)

Examples:

$$V_{12} = -\frac{5120(1+8x+2y+17x^2+17y^2)}{(160+4x^2+4y^2+16x^3+4x^2y+16xy^2+4y^3+17(x^2+y^2)^2)^2}, \tag{6}$$

$$\begin{aligned} u_1 &= \frac{x+2x^2+xy-2y^2}{160+4x^2+4y^2+16x^3+4x^2y+16xy^2+4y^3+17(x^2+y^2)^2}, \\ u_2 &= \frac{2x+2y+3x^2+10xy-3y^2}{160+4x^2+4y^2+16x^3+4x^2y+16xy^2+4y^3+17(x^2+y^2)^2}. \end{aligned} \tag{7}$$

Theorem

V_{12} is smooth and decays as $1/r^6$ for $r \to \infty$.
u_1 and u_2 are smooth and decay as $1/r^2$ for $r \to \infty$ so they lie in the kernel of $L = -\Delta + u : L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$.

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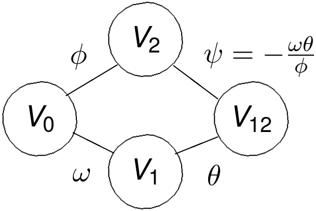
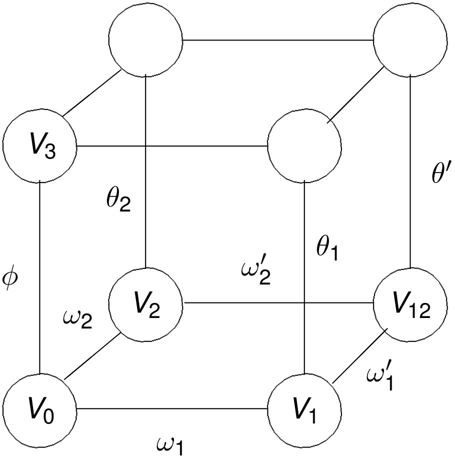
Nonlinear superposition principles

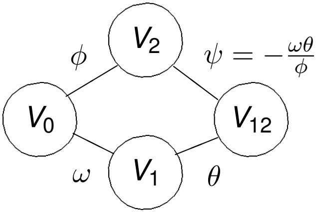


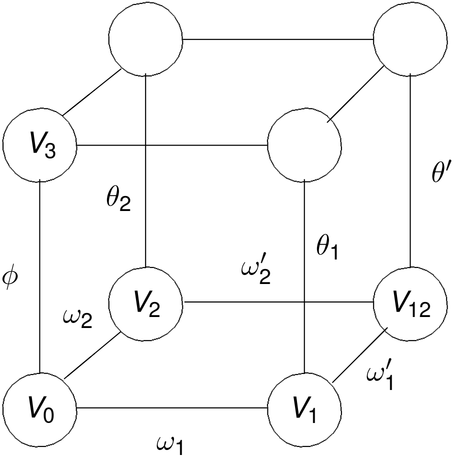


The algebraic superposition formula (L.Bianchi)

$$\theta' = -\phi + \frac{\omega_1\omega_2}{\lambda}(\theta_1 - \theta_2),$$

where $\lambda = \omega_1\omega_1' = -\omega_2\omega_2'$.





Where factorization is hidden???

Example: $L = D_x^2 - D_y^2 - \frac{2}{x^2}$

Laplace transformation: $L \xrightarrow{P} L_1$,
$P = D_x - D_y$,
$L_1 = D_x^2 - D_y^2 + \frac{2}{x}(D_x + D_y) - \frac{2}{x^2}$

$L_1 = (D_x + D_y)(D_x - D_y + \frac{2}{x})$.

In fact L and L_1 are *isomorphic*: $L_1 \xrightarrow{Q} L$,
$Q = (D_x + D_y) - \frac{2}{x^2}$

Another interpretation of this isomorphism:
$L_1 \cdot P = Q_1 \cdot L$,
$L \cdot Q = P_1 \cdot L_1$,

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Attacking the multidimensional case: *n*-dim solvable Schrödinger equations and Huygens' principle

n-dimensional stationary Schrödinger equation:

$$L = -(D_{x_1}^2 + \ldots + D_{x_n}^2) + V(x_1, \ldots, x_n)u + Eu = 0 \qquad (8)$$

"D-integrable potentials" (Berest Yu.Yu., Chalykh O., Veselov A. P. . . .) *for arbitrary E*:

$$V = \sum_{k=1}^{m} \frac{(\alpha_k, \alpha_k)A_k}{((\alpha_k, \vec{x}) + c_k)^2}.$$

Problem: singularities.

Main Ansatz: $L \cdot M = M \cdot L_0$,

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Factorization of *linear* PDEs and *nonlinear* PDEs

$$u_{xy} = F(x, y, u, u_x, u_y), \quad u = u(x, y). \tag{9}$$

Idea: linearization.
$u(x, y) \to u(x, y) + \epsilon v(x, y) \Longrightarrow$

$$v_{xy} = Av_x + Bv_y + Cv \tag{10}$$

Theorem

(Anderson, Juras, Sokolov, 1995) *A second order, scalar, hyperbolic partial differential equation (9) is Darboux integrable if and only if both Laplace sequences for (10) are finite ($\Longleftrightarrow$ it is factorizable in a generalized sense).*

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Problems for LPDEs

Blumberg-Landau (around 1910):

if

$$P = D_x + xD_y, \quad Q = D_x + 1,$$

$$R = D_x^2 + xD_xD_y + D_x + (2+x)D_y,$$

then $L = Q \cdot Q \cdot P = R \cdot Q$.

R is absolutely irreducible, i.e. one can not factor it into product of first-order operators with coefficients in any extension of $Q(x, y)$.

So Jordan-Hölder-Landau theorem does NOT hold for the "naive" definition of factorization as decomposition into a product of lower-order operators in the ring $Q(x, y)[D_x, D_y]$.

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A challenge

$$Lu = (D_x D_y + x D_x D_z - D_z)u = 0.$$

It has a complete solution (U. Dini, 1902):

$$u = \int \Big(v\,dx + (D_y + xD_z)v\,dz \Big) + \theta(y),$$

where $v = \int \phi(x, xy - z)\,dx + \psi(y, z)$.

Technology: *"Dini transformations"*

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Technology: *“Dini transformations”*

$$\begin{cases} D_x u_1 = u_1 + 2u_2 + u_3, \\ D_y u_2 = -6u_1 + u_2 + 2u_3, \\ (D_x + D_y)u_3 = 12u_1 + 6u_2 + u_3. \end{cases}$$

It has the complete explicit solution (S.Ts., ISSAC'2005):

$$\begin{cases} u_1 = 2e^y G(x) + e^x(3F(y) + F'(y)) + \exp\frac{x+y}{2} H(x-y), \\ u_2 = e^y G'(x) + 2e^x F'(y) - 2u_1, \\ u_3 = D_x u_1 + 3u_1 - 2(e^y G'(x) + 2e^x F'(y)), \end{cases}$$

where $F(y)$, $G(x)$ and $H(x-y)$ are three arbitrary functions of one variable each.

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A bit of history

Euler, Laplace, Moutard, Darboux, ...:

$$u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0.$$

Lagrange: hyperbolic equations

$$a_{11}u_{xx} + a_{12}u_{xy} + a_{22}u_{yy} + a_1 u_x + a_2 u_y + a_0 u = 0,$$

$a_{ij} = a_{ij}(x,y)$, $a_{12}^2 - 4a_{11}a_{22} > 0$.

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1998-2005: a “good” definition of factorization, partial algorithms for generalized factorization (S. Ts.)

A paragon theory: factorization of LODEs

(I) Solutions $\Longleftrightarrow$ factorizations:

$L = L_1 \cdot (D - u(x)) \Longleftrightarrow u = \frac{y'}{y}$, $y(x)$ is a solution of $Ly = 0$.

$L = (D - u_1(x)) \cdots (D - u_n(x)) \Longleftrightarrow$ we know all solutions of L.

(II) Jordan-Hölder theorem (Landau theorem) for different factorizations of a given operator:
$L = L_1 \cdots L_k = \overline{L}_1 \cdots \overline{L}_r \Longrightarrow k = r$.

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Goals:

1) to define a notion of factorization with “good” properties:
1a) $\forall$ LPDO $L \approx L_1 L_2 \cdots L_k$ with FINITE k.
In particular D_x should be irreducible …
1b) Preserving the property proved by Landau E. (1902) for LODE: all possible factorizations of a given operator L have the same number of factors in different expansions $L = L_1 \cdots L_k = \overline{L}_1 \cdots \overline{L}_r$ into irreducible factors and the factors $L_s, \overline{L}_p$ are pairwise “similar".

2) Existence of *large* classes of solutions should be related to factorization.

3) Algorithms?

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Hint 1: if we have $L = L_1 L_2 \cdots L_k \Longleftrightarrow$ we have a chain of left principal ideals
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Hint 2: We shall drop the word "principal" (Blumberg's example).

Hint 3: But we shall take *not all* left ideals!
Example: $|D_x\rangle \subset |D_x, D_y^m\rangle \subset |D_x, D_y^{m-1}\rangle \subset \ldots |D_x, D_y\rangle \subset |1\rangle$.

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1) chains will be finite and for a given L they will have the same length:

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2) Irreducible LODO will be still irreducible as LPDEs.

3) for $dim = 2$, $ord = 2$ LODO, factorizable $\Longleftrightarrow$ integrable (with Laplace transformations).

4) Algebraically, the problem is reduced from $Q(x, y)[D_x, D_y]$ to $Q(x, y, D_x)[D_y]$ and/or $Q(x, y, D_y)[D_x]$ (Ore quotients).

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Abelian category of L.O.D.O.:

objects are operators $L = a_0(x)D^n + a_1(x)D^{n-1} + \ldots + a_n(x)$,

morphisms are mappings of solutions with auxiliary operators: $P : L \to M$ iff for every u such that $Lu = 0$, $v = Pu$ gives a solution of M: $Mv = 0$.

Algebraically: $M \cdot P = N \cdot L$.

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$$S : \begin{cases} L_{11}u_1 + \ldots + L_{1s}u_s = 0, \\ \cdots \\ L_{k1}u_1 + \ldots + L_{ks}u_s = 0, \end{cases}$$

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Any abelian category with finite ascending chains satisfies the Jordan-Hölder property.

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The solution: Serre-Grothendieck factorcategory!

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$$L = D_x^2 - D_y^2 - \frac{2}{x^2} = (\mathrm{D}_x - A)(D_x + A)$$

where $A \in Q(x, D_y)$, so L factors in $Q(x, D_y)[D_x]$!!!

so transition from $Q(x)[D_x, D_y]$ to $Q(x, D_y)[D_x]$
(or $Q(x, D_x)[D_y]$) is *useful*.

Latest result (2011): Dini transformations (in R^3) are Laplace transformations in $Q(x, y, z, D_z)[D_x, D_y]$!

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