Polynomial dynamical systems and heat equations.

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The talk is devoted to a construction that takes a homogeneous polynomial system in an n-dimensional space to a solution of the heat equation, in terms of a solution of an ordinary differential equation of order $(n+1)$.

The theory of elliptic and multidimensional sigma-functions gives us a big amount of such dynamical systems.

Using the Cole-Hopf transformation and our solutions of the heat equation, we obtain the corresponding solutions of the Burgers equation.

In the case of the elliptic sigma function our construction reduces the solution of the heat equation to the solution of an ordinary differential equation of the third order, namely the Chazy equation.

We describe a deformation of the Weierstrass sigma function such that for this function our construction gives the famous Chazy family of third order differential equations.

Results presented in the talk were obtained in recent joint works with E. Yu. Bunkova.
Main definitions will be introduced during the talk.

The standard Weierstrass model for a plane elliptic curve is

$$V = \{(\lambda, \mu) \in \mathbb{C}^2 : \mu^2 = 4\lambda^3 - g_2\lambda - g_3\}. \tag{1}$$

The discriminant of this curve is $\Delta = g_2^3 - 27g_3^2$.
The curve is non-degenerate when $\Delta \neq 0$.

Set

$$2\omega_k = \oint_{a_k} \frac{d\lambda}{\mu}, \qquad 2\eta_k = -\oint_{a_k} \frac{\lambda d\lambda}{\mu}, \qquad k = 1, 2, \tag{2}$$

where $\frac{d\lambda}{\mu}$ and $\frac{\lambda d\lambda}{\mu}$ are the basic differentials
and a_k are the basic cycles on the curve such that

$$\eta_1\omega_2 - \omega_1\eta_2 = \frac{\pi i}{2}.$$

A plane non-degenerate algebraic curve V defines a lattice $\Gamma \subset \mathbb{C}$ of rank 2 generated by $2\omega_1$, $2\omega_2$, with $\mathbf{Im}\frac{\omega_2}{\omega_1} > 0$.

An **elliptic function** is a meromorphic function on $\mathbb{C}$ such that

$$f(z + 2\omega_k) = f(z), \quad k = 1, 2,$$

that is it can be considered
as a function on a **complex torus** $\mathbb{T} = \mathbb{C}/\Gamma$.

The torus $\mathbb{T}$ is known as the **Jacobian** of the curve V.

The Weierstrass $\wp$-function is the unique elliptic function $\wp(z) = \wp(z; g_2, g_3)$ on $\mathbb{C}$ with poles only in lattice points such that $\lim_{z\to 0}\left(\wp(z) - \frac{1}{z^2}\right) = 0$.

It is an even function and all its poles are double poles.
It defines a uniformization of the standard elliptic curve:

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

The function $u(z) = 2\wp(z)$ is a 2-periodic solution of the stationary KdV equation

$$u''' = 6uu'.$$

The Weierstrass ζ-function is the odd meromorphic function $\zeta(z) = \zeta(z; g_2, g_3)$ such that

$$\zeta'(z) = -\wp(z) \qquad \text{and} \qquad \lim_{z\to 0}\left(\zeta(z) - \frac{1}{z}\right) = 0.$$

The periodic properties are

$$\zeta(z + 2\omega_k) = \zeta(z) + 2\eta_k,$$

and we have $\eta_k = \zeta(\omega_k)$, $k = 1, 2$.

The Weierstrass-Stickelberger equation is

$$(\zeta(z_1)+\zeta(z_2)+\zeta(z_3))^2=\wp(z_1)+\wp(z_2)+\wp(z_3)$$

for $z_1+z_2+z_3=0$.

This functional equation led to exact results for a quantum one-dimensional many-body problem of n identical particles with pair interactions. It is the problem of solving the Schrödinger equation

$$-\Delta\Psi_0+U\Psi_0=E_0\Psi_0, \quad \text{where} \quad U=\sum_{1\leqslant i<j\leqslant n} u(x_i-x_j).$$

see F. Calogero, *One-dimensional many-body problems with pair interactions whose exact ground-state wave function is of product type*, Lett. Nuovo Cimento 13 507-511, **1975**
B. Sutherland, *Exact ground-state wave function for a one-dimensional plasma*, Phys. Rev. Lett. 34 1083-1085, **1975**.

The Weierstrass σ-function is the entire **odd** meromorphic function $\sigma(z) = \sigma(z; g_2, g_3)$ such that

$$\Big(\ln \sigma(z)\Big)' = \zeta(z) \qquad \text{and} \qquad \lim_{z \to 0} \left(\frac{\sigma(z)}{z}\right) = 1.$$

The periodic properties are

$$\sigma(z + 2\omega_k) = -\sigma(z) \exp\Big(2\eta_k (z + \omega_k)\Big), \quad k = 1, 2.$$

There is the equation

$$\frac{\sigma(z_1+z_2)\sigma(z_1-z_2)}{\sigma(z_1)^2\sigma(z_2)^2} = \wp(z_2) - \wp(z_1).$$

This functional equation led to integration of the equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + 2\sum_{i=1}^{n} \wp(x - x_i(t))\right)\Psi = 0,$$

see I.M. Krichever,

Elliptic solutions of the Kadomtsev-Petviashvili equation and integrable systems of particles,

Functional Analysis and Its Applications, 1980, 14:4, 282–290.

In the case $\Delta = 0$ one can find γ such that $g_2 = \frac{4}{3}\gamma^4$, $g_3 = \frac{8}{27}\gamma^6$. Then

$$\sigma(z; \frac{4}{3}\gamma^4, \frac{8}{27}\gamma^6) = \frac{1}{\gamma}\exp(\frac{1}{6}\gamma^2 z^2)\sin\gamma z,$$

$$\zeta(z; \frac{4}{3}\gamma^4, \frac{8}{27}\gamma^6) = \frac{1}{3}\gamma^2 z + \gamma\,\mathrm{ctg}\,\gamma z,$$

$$\wp(z; \frac{4}{3}\gamma^4, \frac{8}{27}\gamma^6) = \gamma^2(-\frac{1}{3} + \frac{1}{(\sin\gamma z)^2}).$$

In the general case there exists a smooth parameter δ such that $g_2 \to \frac{4}{3}\gamma^4$, $g_3 \to \frac{8}{27}\gamma^6$ for $\delta \to 0$.

Consider the fields on $\mathbb{C}^2$

$$l_0 = 4g_2\frac{\partial}{\partial g_2} + 6g_3\frac{\partial}{\partial g_3}, \quad l_2 = 6g_3\frac{\partial}{\partial g_2} + \frac{1}{3}g_2^2\frac{\partial}{\partial g_3}.$$

We have $[l_0, l_2] = 2l_2, \quad l_0\Delta = 12\Delta, \quad l_2\Delta = 0, \quad \langle l_0, l_2\rangle = \frac{4}{3}\Delta.$

Let τ_0 be the parameter on the family of curves defined by the dynamical system ($l_0 = \frac{\partial}{\partial \tau_0}$)

$$g_2' = 4g_2, \quad g_3' = 6g_3.$$

Then $g_2(\tau_0) = g_2(0)e^{4\tau_0}$, $g_3(\tau_0) = g_3(0)e^{6\tau_0}$.

Let τ_2 be the parameter on the family of curves defined by the dynamical system ($l_2 = \frac{\partial}{\partial \tau_2}$)

$$g_2' = 6g_3, \quad g_3' = \frac{1}{3}g_2^2.$$

It is a homogeneous polynomial dynamical system with $\deg \tau_2 = 4$, $\deg g_k = -4k$, $k = 2, 3$.
Then

$$g_2(\tau_2) = 3\wp(\tau_2 + d; 0, b_3), \qquad g_3(\tau_2) = \frac{1}{2}\wp'(\tau_2 + d; 0, b_3),$$

where b_3 is defined by the initial data as $b_3 = \frac{4}{27}g_2(0)^3 - 4g_3(0)^2$, and d is the solution of the compatible system of equations $\wp(d; 0, b_3) = \frac{1}{3}g_2(0)$, $\wp'(d; 0, b_3) = 2g_3(0)$.

Let

$$l_0 = 4g_2\frac{\partial}{\partial g_2} + 6g_3\frac{\partial}{\partial g_3}, \qquad l_2 = 6g_3\frac{\partial}{\partial g_2} + \frac{1}{3}g_2^2\frac{\partial}{\partial g_3},$$

$$H_0 = z\frac{\partial}{\partial z} - 1, \qquad H_2 = \frac{1}{2}\frac{\partial^2}{\partial z^2} + \frac{1}{24}g_2 z^2,$$

$$Q_0 = H_0 - l_0, \qquad Q_2 = H_2 - l_2.$$

The Weierstrass theorem.

The operators Q_0 and Q_2 annihilate the sigma-function:

$$Q_0\sigma(z; g_2, g_3) = 0, \qquad Q_2\sigma(z; g_2, g_3) = 0.$$

Theorem. The function $\psi(z,t)$ such that

$$\psi(z,t) = e^{h(t)z^2+r(t)}\sigma\left(z, g_2(t), g_3(t)\right) \tag{3}$$

for some functions $r(t)$, $h(t)$, $g_2(t)$ and $g_3(t)$ satisfies the heat equation

$$\frac{\partial}{\partial t}\psi(z,t) = \frac{1}{2}\frac{\partial^2}{\partial z^2}\psi(z,t) \tag{4}$$

if and only if the functions $r(t)$, $h(t)$, $g_2(t)$ and $g_3(t)$ satisfy the homogeneous polynomial dynamical system in $\mathbb{C}^4$ with coordinates (h, r, g_2, g_3), $\deg h = -4$, $\deg r = 0$:

$$h' = 2h^2 - \frac{1}{24}g_2, \qquad r' = 3h, \tag{5}$$

$$g_2' = 6g_3 + 8hg_2, \qquad g_3' = \frac{1}{3}g_2^2 + 12hg_3. \tag{6}$$

Theorem. The functions $r(t)$, $h(t)$, $g_2(t)$ and $g_3(t)$ satisfy the dynamical system (5) - (6) if and only if $h(t)$ satisfies the Chazy equation

$$h''' - 24hh'' + 36(h')^2 = 0, \tag{7}$$

and

$$g_2 = -24(h' - 2h^2), \quad g_3 = -4(h'' - 12h'h + 16h^3), \quad r' = 3h.$$

For initial data

$$h_0 = h(0), \quad h_1 = h'(0), \quad h_2 = h''(0)$$

there exists a unique solution of the Chazy equation (7).

Corollary. For given (h_0, h_1, h_2) there exists a unique up to a factor solution of the heat equation (4) of the form (3).

Example. The function

$$\psi(z,t) = \exp(-2\gamma^2 t)\frac{\sin\gamma z}{\gamma}, \quad \gamma = const$$

is a **periodic odd** function of z.
It is the classical solution of the heat equation.

In this case we have

$$r = -\frac{1}{2}\gamma^2 t, \quad h = -\frac{1}{6}\gamma^2, \quad g_2 = \frac{4}{3}\gamma^4, \quad g_3 = \frac{8}{27}\gamma^6.$$

Example.

For the classical solution

$$\psi(z,t) = \psi_*(z,t) - \psi_*(-z,t),$$

where

$$\psi_*(z,t) = \frac{1}{\sqrt{t}} \exp\left(-\frac{(z-a)^2}{2t}\right),$$

which is decreasing when $z \to \pm\infty$, we have $\gamma = -\frac{ia}{t}$ and

$$h = \frac{a^2 - 3t}{6t^2}, \quad r = \ln(\frac{2a}{t^3}) - \frac{a^2}{2t}, \quad g_2 = \frac{4}{3}\frac{a^4}{t^4}, \quad g_3 = -\frac{8}{27}\frac{a^6}{t^6}.$$

Let $\psi(z,t)$ be an **odd** function of z **regular** at $z=0$.

Such a function can be uniquely represented in the form

$$\psi(z,t) = e^{h(t)z^2+r(t)}\phi(z,t), \tag{8}$$

where the function $\phi(z,t)$ in the vicinity of $z=0$ is given by the series

$$\phi(z,t) = z + \sum_{k\geqslant 2} \phi_k(t)\frac{z^{2k+1}}{(2k+1)!}.$$

Theorem. The function $\psi(z,t)$ in the vicinity of $z=0$ is a solution to the heat equation

$$\frac{\partial\psi}{\partial t}=\frac{1}{2}\frac{\partial^2\psi}{\partial z^2}$$

if and only if $r'=3h$ and the function $\phi(z,t)$ is a solution to

$$\frac{\partial\phi}{\partial t}=\mathcal{H}_2\phi+2h\mathcal{H}_0\phi,$$

where

$$\mathcal{H}_2\phi=\left(\frac{1}{2}\frac{\partial^2}{\partial z^2}+\frac{1}{24}uz^2\right)\phi,\quad \mathcal{H}_0\phi=(z\frac{\partial}{\partial z}-1)\phi,\quad u=-24(h'-2h^2).$$

Using this theorem we search for solutions of the form (8) of the heat equation.

Put $\deg z = 2$, $\deg t = 4$ and

$$\phi_k(t) = \Phi_k(x_2(t), \ldots, x_{n+1}(t))$$

where Φ_k are homogeneous polynomials of degree $-4k$ with $\deg x_q = -4q$.

With such assumptions we have

$$\mathcal{H}_0 \phi = -\delta_0 \phi,$$

where

$$2\delta_0 = \sum_{k=2}^{n+1} (-4k) x_k \frac{\partial}{\partial x_k}.$$

Consider in $\mathbb{C}^n$ with coordinates $(x_2, \ldots, x_{n+1})$, $\deg x_q = -4q$ some homogeneous dynamical system of τ

$$\frac{\partial x_k}{\partial \tau} = p_{k+1}(x_2, \ldots, x_{n+1}),$$

where $\deg p_q = -4q$, and set

$$\delta_2 \phi = \sum p_{k+1}(x_2, \ldots, x_{n+1}) \frac{\partial}{\partial x_k} \phi.$$

Theorem. The function

$$\Phi(z; x_2, \ldots, x_{n+1}) = z + \sum_{k \geqslant 2} \Phi_k(x_2, \ldots, x_{n+1}) \frac{z^{2k+1}}{(2k+1)!} \tag{9}$$

gives a solution to the equation

$$\mathcal{H}_2 \Phi = \delta_2 \Phi$$

if and only if

$$\Phi_2(x_2, \ldots, x_{n+1}) = -\frac{1}{2} u, \qquad \Phi_3 = 2 \frac{\partial}{\partial \tau} \Phi_2,$$

and

$$\Phi_{k+1} = 2 \frac{\partial}{\partial \tau} \Phi_k + \frac{1}{3} k(2k+1) \Phi_2 \Phi_{k-1}, \quad k > 2.$$

Let $u = x_2$. Thus for the homogeneous system we get

$$\Phi_2 = -\frac{1}{2} x_2,$$

and the function (9) gives a solution to the equations

$$\mathcal{H}_0 \Phi = -\delta_0 \Phi, \qquad \mathcal{H}_2 \Phi = \delta_2 \Phi$$

where

$$2\delta_0 = -\sum 4k x_k \frac{\partial}{\partial x_k}, \quad \delta_2 = \sum p_{k+1}(x_2, \dots, x_{n+1}) \frac{\partial}{\partial x_k}.$$

In the equation

$$\frac{\partial}{\partial t}\phi = \mathcal{H}_2\phi + 2h\mathcal{H}_0\phi$$

for the ansatz

$$\phi(z,t) = \Phi(z, x_1(t), \ldots, x_{n+1}(t))$$

we have

$$\sum_{k=2}^{n+1}\left(\frac{\partial x_k}{\partial t} - p_{k+1}(x_2, \ldots, x_{n+1}) - 4hkx_k\right)\frac{\partial \Phi}{\partial x_k} = 0.$$

In the case of a non-degenerate function Φ, the functions $\frac{\partial \Phi}{\partial x_k}$ are lineary independant and we have

$$\frac{\partial x_k}{\partial t} = p_{k+1}(x_2, \ldots, x_{n+1}) + 4khx_k.$$

Corollary.

In the space $\mathbb{C}^{n+1}$ with the coordinates $x_1 = h$, x_2, $\ldots x_{n+1}$ we get the homogeneous polynomial dynamical system

$$\frac{\partial}{\partial t} h = 2h^2 - \frac{1}{24} x_2,$$

$$\frac{\partial}{\partial t} x_k = p_{k+1}(x_2, \ldots, x_{n+1}) + 4khx_k, \quad k = 2, \ldots, n+1.$$

Applications of general construction.

Example $n = 2$.
Consider the homogeneous dynamical system:

$$\frac{\partial x_2}{\partial \tau} = 6x_3, \qquad \frac{\partial x_3}{\partial \tau} = \frac{1}{3}x_2^2.$$

We have $(x_2')^2 = \frac{4}{3}x_2^3 - 9b_3$ for some constant b_3,
where $\deg x_k = -4k$, $\deg b_3 = -24$. Thus for some constant d

$$x_2 = 3\wp(\tau + d; 0, b_3), \quad x_3 = \frac{1}{2}\wp'(\tau + d; 0, b_3).$$

We get the dynamical system

$$\frac{\partial x_2}{\partial t} = 6x_3 + 8hx_2, \quad \frac{\partial x_3}{\partial t} = \frac{1}{3}x_2^2 + 12hx_3, \quad \frac{\partial h}{\partial t} = 2h^2 - \frac{1}{24}x_2.$$

which leads to the Chazy equation:

$$h''' - 24hh'' + 36(h')^2 = 0.$$

Example $n = 3$**.** Consider the homogeneous dynamical system:

$$\frac{\partial x_2}{\partial \tau} = 6x_3, \quad \frac{\partial x_3}{\partial \tau} = \frac{1}{3}x_2^2 + 2\epsilon x_4, \quad \frac{\partial x_4}{\partial \tau} = 2\gamma x_2 x_3.$$

Here ϵ and γ are scalars and $\deg \epsilon = \deg \gamma = 0$.
From this system we have

$$x_2 = \frac{3}{1+\epsilon\gamma}\wp(\tau + d; \epsilon b_2, b_3), \quad x_3 = \frac{1}{2+2\epsilon\gamma}\wp'(\tau + d; \epsilon b_2, b_3)$$

where b_2, b_3 and d are some constants. We get the system

$$\frac{\partial x_2}{\partial t} = 6x_3 + 8hx_2, \qquad \frac{\partial x_3}{\partial t} = \frac{1}{3}x_2^2 + 2\epsilon x_4 + 12hx_3,$$
$$\frac{\partial x_4}{\partial t} = 2\gamma x_2 x_3 + 16hx_4, \qquad \frac{\partial h}{\partial t} = 2h^2 - \frac{1}{24}x_2,$$

which leads to the equation:

$$C(h)' - 16hC(h) - 96\gamma\epsilon(2h^2 - h')(h'' - 12hh' + 16h^3) = 0.$$

Here $C(h) = h''' - 24hh'' + 36(h')^2$.

The equation

$$C(h)' - 16hC(h) - 96\gamma\epsilon(2h^2 - h')(h'' - 12hh' + 16h^3) = 0$$

with $C(h) = h''' - 24hh'' + 36(h')^2$ is equivalient to the equation

$$\widehat{\mathrm{C}}(h)' - 16h\widehat{\mathrm{C}}(h) = 0$$

with

$$\widehat{\mathrm{C}}(h) = C(h) + 48\gamma\epsilon\left((h')^2 - 4h'h^2 + 4h^4\right).$$

Corollary.
Let $y(t) = 12h(t)$, $\alpha = -\frac{1}{9}\gamma\epsilon$. Then

$$12\widehat{\mathrm{C}}(h) = y''' - 2yy'' + 3(y')^2 - \alpha(6y' - y^2)^2.$$

In the study of third order ordinary differential equations having **the Painlevé property** Chazy was led to the remarkable family of equations

$$y''' = 2yy'' - 3(y')^2 + \alpha(6y' - y^2)^2,$$

see J. Chazy,
Sur les équations différentielles du troisième ordre et d'ordre supérieur dont l'intégrale générale a ses points critiques fixes, Acta Math. 34, 317-385, **1911**.

The importance of the Chazy family in the modern theory of integrable systems is discribed in
Clarkson, P.A., and Olver, P.J., *Symmetry and the Chazy equation*, J. Diff. Eq. 124, 225-246, **1996**.

The solutions of the Burgers equation.

Consider the Burgers equation

$$v_t + vv_z = \frac{1}{2}v_{zz}. \tag{10}$$

The Cole-Hopf transform of a function $\psi(z,t)$ is

$$v(z,t) = -\frac{\partial \ln \psi(z,t)}{\partial z}.$$

There is the identity

$$v_t + vv_z - \frac{1}{2}v_{zz} = -\frac{\partial}{\partial z}\left(\frac{\psi_t - \frac{1}{2}\psi_{zz}}{\psi}\right).$$

Corollary. Let $\psi(z,t)$ be a solution of the heat equation. Then $v(z,t)$ is a solution of the Burgers equation (10).

The elliptic case.
Let

$$\psi(z,t) = e^{h(t)z^2 - r(t)}\sigma\left(z, g_2(t), g_3(t)\right),$$

as before be the solution of the heat equation

$$\frac{\partial}{\partial t}\psi(z,t) = \frac{1}{2}\frac{\partial^2}{\partial z^2}\psi(z,t).$$

Theorem. The function

$$v(z,t) = -\frac{\partial \ln \psi(z,t)}{\partial z} = -2h(t)z - \zeta(z; g_2(t), g_3(t))$$

gives a solution of the Burgers equation,
where $h(t)$ is the solution of the Chazy equation

$$h''' - 24hh'' + 36(h')^2 = 0,$$

and g_2, g_3, h are given by

$$g_2 = 24(2h^2 - h'), \quad g_3 = -4(h'' - 12h'h + 16h^3).$$

Corollary.
Fix any h_0, $g_{2,0}$, $g_{3,0}$. A solution of the Chazy equation with

$$h(t_0) = h_0, \quad h'(t_0) = -\frac{1}{24}g_{2,0} + 2h_0^2, \quad h''(t_0) = -\frac{1}{4}g_{3,0} - \frac{1}{2}h_0 g_{2,0} + 8h_0^3$$

provides a solution $v(z,t)$ of the Burgers equation with the initial wave $v(z,t_0) = -2h_0 z - \zeta(z; g_{2,0}, g_{3,0})$.

For the classical periodic of z solution of the heat equation we have a stationary solution of the Burgers equation

$$v(z,t) = -\gamma \operatorname{ctg}(\gamma z), \quad \gamma = const.$$

Corollary. For the solution of the heat equation

$$\psi_*(z,t) = \frac{1}{\sqrt{t}} \exp(-\frac{(z-a)^2}{2t})$$

we get the non-stationary solution of the Burgers equation

$$v(z,t) = \frac{1}{t}(z-a).$$

Corollary. For the solution of the heat equation

$$\psi(z,t) = \frac{1}{\sqrt{t}} \exp(-\frac{z^2+a^2}{2t}) \left(\exp(\frac{az}{t}) - \exp(\frac{-az}{t})\right)$$

we get the non-stationary solution of the Burgers equation

$$v(z,t) = \frac{1}{t}(z - a \operatorname{cth} \frac{az}{t}).$$

Properties of the solutions.
Let $v(z,t)$ be a solution of the Burgers equation discribed before. It is an odd function of z and

$$v(z+2\omega_k(t),t) = v(z,t) + 2v(\omega_k(t),t), \qquad k = 1,2.$$

General lemma. Let $h(t)$ be a solution of the Chazy equation. The function

$$(Th)(t) = \frac{1}{(ct+d)^2} h\left(\frac{at+b}{ct+d}\right) - \frac{c}{2(ct+d)}, \qquad |c| + |d| > 0,$$

satisfies the Chazy equation if and only if $(ad-bc-1)(ad-bc) = 0$.

Corollary. The solution $(Tv)(z,t)$ of the Burgers equation corresponding to $(Th)(t)$ is defined.

For $h(t) \equiv 0$ we get $g_2 = g_3 = 0$ and $v(z,t) = -\frac{1}{z}$ is a solution of the stationary Burgers equation $vv_z = \frac{1}{2}v_{zz}$.

Example. For $h(t) \equiv 0$ we have

$$(Th)(t) = -\frac{c}{2(ct+d)}.$$

We get $g_2 = g_3 = 0$ and the solution of the Burgers equation

$$(Tv)(z,t) = -\frac{1}{z} + \frac{cz}{(ct+d)}.$$

For $h(t) \equiv k = const$ we get $g_2 = 48k^2$, $g_3 = -64k^3$, and the solution of the stationary Burgers equation

$$v(z,t) = -\sqrt{-6k}\,\mathrm{ctg}(\sqrt{-6k}z).$$

Example. For $(Th)(t) = \frac{k}{(ct+d)^2} - \frac{c}{2(ct+d)}$ where $h(t) \equiv k = const$:

$$g_2(t) = 48\frac{k^2}{(ct+d)^4}, \qquad g_3(t) = -64\frac{k^3}{(ct+d)^6},$$

whence $\Delta = 0$, and for $\gamma(t) = \frac{\sqrt{6k}}{(ct+d)}$ a solution of the Burgers equation is

$$(Tv)(z,t) = \gamma\left(\frac{c}{\sqrt{6k}}z - \mathrm{cth}\,\gamma z\right)$$